Chapter 7

Programming Language Semantics

Overview of Chapter

Section 7.1

Introduction

Motivation

Why studying language semantics?

- Understanding the details and construction of programming and modeling languages
- Foundation for language processing tools (compilers, optimizers, interpreters,...)
- Verifying type systems
- Development of new language abstractions and software concepts
- Reasoning about software

Material

Literature

- Glynn Winskel:
  The Formal Semantics of Programming Languages: An Introduction
- Benjamin C. Pierce et al.:
  Software Foundations (`www.cis.upenn.edu/~bcpierce/sf/`)

Acknowledgement

Thanks to Prof. Peter Müller for the slides.

General aspects

Degree of formalization:

- Informal semantics in language reports
- Formalization to support proof tools and as quality control

Goals here:

- Learn distinction between operational and denotational semantics
- Learn about the formalization of operational semantics
- Understand the relationship between programming language semantics and transition systems

Why Formal Semantics?

- Programming language design
  - Formal verification of language properties
  - Reveal ambiguities
  - Support for standardization
- Implementation of programming languages
  - Compilers
  - Interpreters
  - Portability
- Reasoning about programs
  - Formal verification of program properties
  - Extended static checking

Language Properties

- Type safety:
  In each execution state, a variable of type T holds a value of T or a subtype of T
- Very important question for language designers
- Example:
  If String is a subtype of Object, should `String[]` be a subtype of `Object[]`?

Language Properties

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- Example:
  If String is a subtype of Object, should `String[]` be a subtype of `Object[]`?

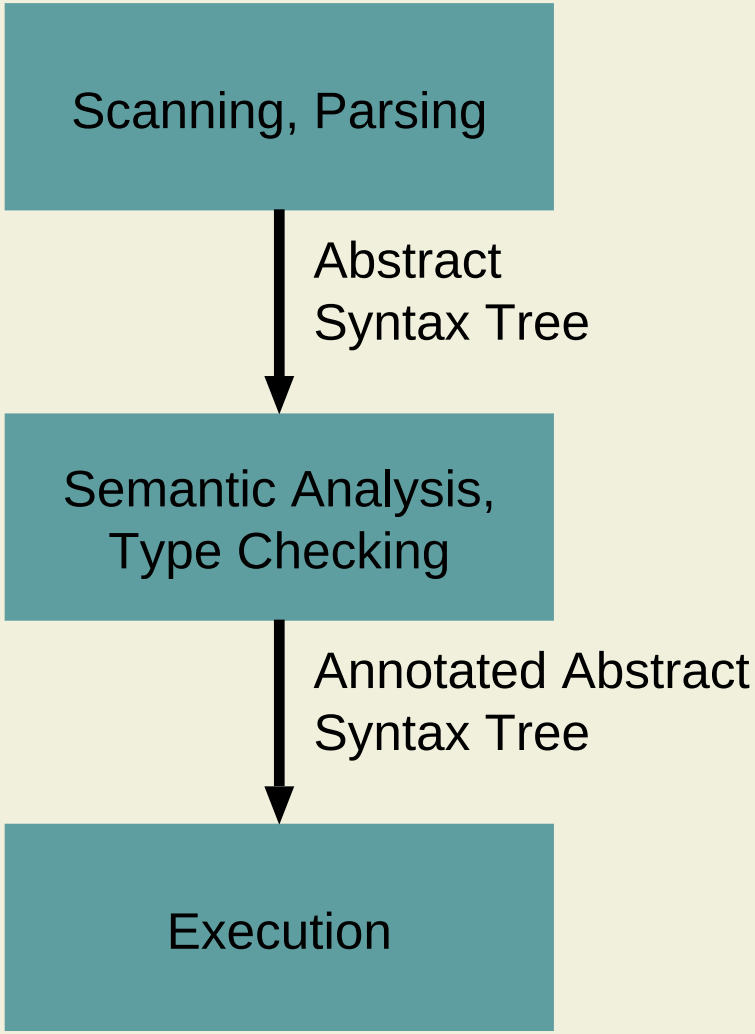
```
void m(Object[] oa) {
  oa[0]=new Integer(5);
}
```

```
String[] sa=new String[10];
m(sa);
String s = sa[0];
```

Language Definition

Dynamic Semantics	▸ State of a program execution ▸ Transformation of states
Static Semantics	▸ Type rules ▸ Name resolution
Syntax	▸ Syntax rules, defined by grammar

Compilation and Execution



Three Kinds of Semantics

- Operational semantics
  - Describes execution on an **abstract machine**
  - Describes **how** the effect is achieved
- Denotational semantics
  - Programs are regarded as **functions** in a mathematical domain
  - Describes **only the effect**, not how it is obtained
- Axiomatic semantics
  - **Specifies properties** of the effect of executing a program are expressed
  - Some aspects of the computation may be **ignored**

Operational Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

- "First we assign 1 to y, then we test whether x is 1 or not. If it is then we stop and otherwise we update y to be the product of x and the previous value of y and then we decrement x by 1. Now we test whether the new value of x is 1 or not. . . "
- Two kinds of operational semantics
  - Natural Semantics
  - Structural Operational Semantics

Denotational Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

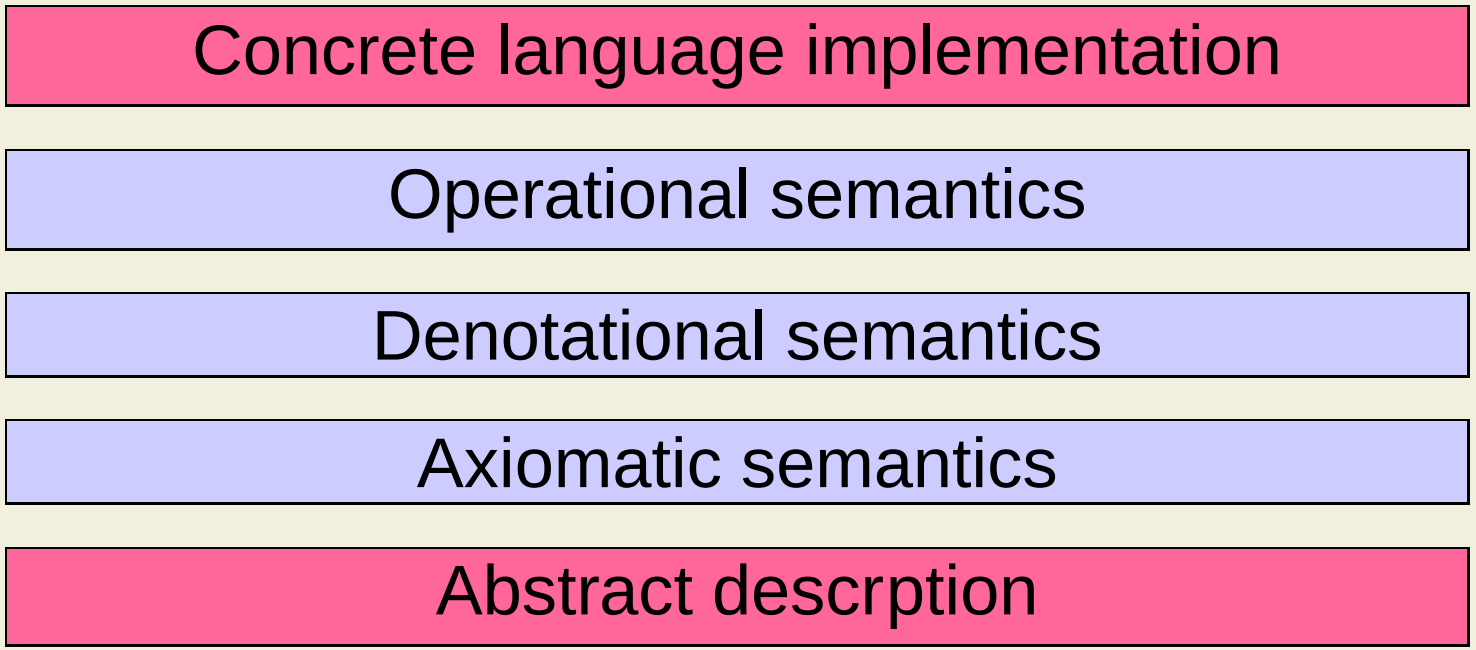
- "The program computes a partial function from states to states: the final state will be equal to the initial state except that the value of x will be 1 and the value of y will be equal to the factorial of the value of x in the initial state"
- Two kinds of denotational semantics
  - Direct Style Semantics
  - Continuation Style Semantics

Axiomatic Semantics

```
y := 1;
while not(x=1) do ( y := x*y; x := x-1 )
```

- "If x$= n$ holds before the program is executed then y$= n!$ will hold when the execution terminates (if it terminates)"
- Two kinds of axiomatic semantics
  - Partial correctness
  - Total correctness

Abstraction



Selection Criteria

- Constructs of the programming language
  - Imperative
  - Functional
  - Concurrent
  - Object-oriented
  - Non-deterministic
  - Etc.
- Application of the semantics
  - Understanding the language
  - Program verification
  - Prototyping
  - Compiler construction
  - Program analysis
  - Etc.

The Language IMP

- Expressions
  - Boolean and arithmetic expressions
  - No side-effects in expressions
- Variables
  - All variables range over integers
  - All variables are initialized
  - No global variables
- IMP does not include
  - Heap allocation and pointers
  - Variable declarations
  - Procedures
  - Concurrency

Syntax of IMP: Characters and Tokens

Characters

```
Letter  = 'A' ... 'Z' | 'a' ... 'z'
Digit   = '0' | '1' | '2' | '3' | '4' | '5' | '6' | '7' | '8' | '9'
```

Tokens

```
Ident   = Letter { Letter | Digit }
Integer = Digit { Digit }
Var     = Ident
```

Syntax of IMP: Expressions

Arithmetic expressions

```
Aexp  = Aexp Op Aexp | Var | Integer
Op    = '+' | '-' | '*' | '/' | 'mod'
```

Boolean expressions

```
Bexp    = Bexp 'or' Bexp | Bexp 'and' Bexp
        | 'not' Bexp | Aexp RelOp Aexp
RelOp   = '=' | '#' | '<' | '<=' | '>' | '>='
```

Syntax of IMP: Statemens

```
Stm  = 'skip'
     | Var ':=' Aexp
     | Stm ';' Stm
     | 'if' Bexp 'then' Stm 'else' Stm 'end'
     | 'while' Bexp 'do' Stm 'end'
```

Notation

Meta-variables (written in *italic* font)

x, y, z	for variables (Var)
e, e', e_1, e_2	for arithmetic expressions (Aexp)
b, b_1, b_2	for boolean expressions (Bexp)
s, s', s_1, s_2	for statements (Stm)

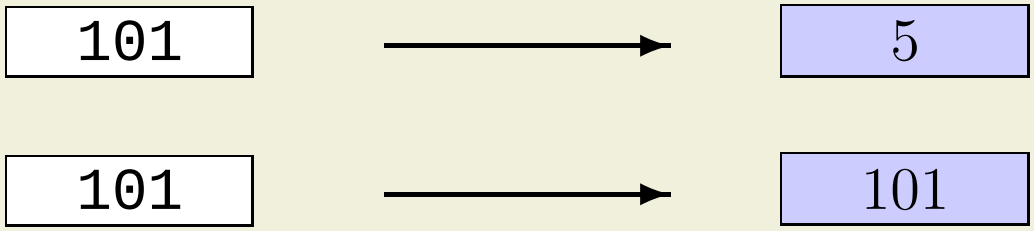
Keywords are written in `typewriter` font

Syntax of IMP: Example

```
res := 1;
while n > 1 do
  res := res * n;
  n := n - 1
end
```

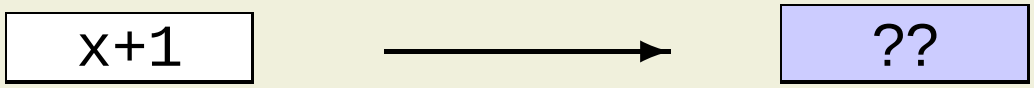
Semantic Categories

Syntactic category: Integer Semantic category: Val $= \mathbb{Z}$



- Semantic functions map elements of syntactic categories to elements of semantic categories
- To define the semantics of IMP, we need semantic functions for
  - Arithmetic expressions (syntactic category Aexp)
  - Boolean expressions (syntactic category Bexp)
  - Statements (syntactic category Stm)

States



- The meaning of an expression depends on the values bound to the variables that occur in it
- A state associates a value to each variable

$$\text{State} : \text{Var} \to \text{Val}$$

- We represent a state σ as a finite function

$$\sigma = \{x_1 \mapsto v_1, x_2 \mapsto v_2, \ldots, x_n \mapsto v_n\}$$

where $x_1, x_2, \ldots, x_n$ are different elements of Var and $v_1, v_2, \ldots, v_n$ are elements of Val.

Semantics of Arithmetic Expressions

The semantic function

$$\mathcal{A} : \text{Aexp} \to \text{State} \to \text{Val}$$

maps an arithmetic expression e and a state σ to a value $\mathcal{A}[\![e]\!]\sigma$

$$\begin{array}{lll} \mathcal{A}[\![x]\!]\sigma & = \sigma(x) & \\ \mathcal{A}[\![i]\!]\sigma & = i & \text{for } i \in \mathbb{Z} \\ \mathcal{A}[\![e_1\ op\ e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma & \text{for } op \in \text{Op} \end{array}$$

$\overline{op}$ is the operation $\text{Val} \times \text{Val} \to \text{Val}$ corresponding to op

Semantics of Boolean Expressions

The semantic function

$$\mathcal{B} : \text{Bexp} \to \text{State} \to \text{Bool}$$

maps a boolean expression b and a state σ to a truth value $\mathcal{B}[\![b]\!]\sigma$

$$\mathcal{B}[\![e_1\ op\ e_2]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma \\ ff & \text{otherwise} \end{cases}$$

$op \in \text{RelOp}$ and $\overline{op}$ is the relation $\text{Val} \times \text{Val}$ corresponding to op

Boolean Expressions (cont'd)

$$\mathcal{B}[\![b_1 \text{ or } b_2]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{B}[\![b_1]\!]\sigma = tt \text{ or } \mathcal{B}[\![b_2]\!]\sigma = tt \\ ff & \text{otherwise} \end{cases}$$

$$\mathcal{B}[\![b_1 \text{ and } b_2]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{B}[\![b_1]\!]\sigma = tt \text{ and } \mathcal{B}[\![b_2]\!]\sigma = tt \\ ff & \text{otherwise} \end{cases}$$

$$\mathcal{B}[\![\texttt{not } b]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{B}[\![b]\!]\sigma = ff \\ ff & \text{otherwise} \end{cases}$$

Operational Semantics of Statements

- Evaluation of an expression in a state yields a value

```
x + 2 * y
```

$\mathcal{A}$: Aexp $\rightarrow$ State $\rightarrow$ Val

- Execution of a statement modifies the state

```
x := 2 * y
```

- Operational semantics describe **how** the state is modified during the execution of a statement

Big-Step and Small-Step Semantics

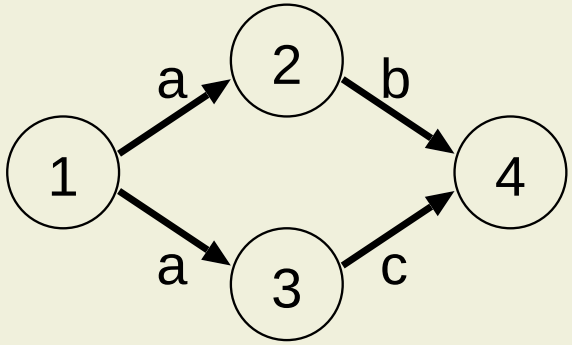
- Big-step semantics describe how the **overall** results of the executions are obtained
  - Natural semantics
- Small-step semantics describe how the **individual steps** of the computations take place
  - Structural operational semantics
  - Abstract state machines

Transition Systems

- A transition system is a tuple $(\Gamma, T, \triangleright)$
  - Γ: a set of **configurations**
  - T: a set of **terminal configurations**, $T \subseteq \Gamma$
  - $\triangleright$: a **transition relation**, $\triangleright \subseteq \Gamma \times \Gamma$

- Example: Finite automaton

$$\begin{aligned}
\Gamma &= \{\langle w, S\rangle \mid w \in \{a,b,c\}^*, S \in \{1,2,3,4\}\} \\
T &= \{\langle \epsilon, S\rangle \mid S \in \{1,2,3,4\}\} \\
\triangleright &= \{(\langle aw, 1\rangle \to \langle w, 2\rangle), (\langle aw, 1\rangle \to \langle w, 3\rangle), \\
&\quad (\langle bw, 2\rangle \to \langle w, 4\rangle), (\langle cw, 3\rangle \to \langle w, 4\rangle)\}
\end{aligned}$$



Section 7.2

Big-step semantics

Big-step semantics

Introductory remarks:

- Big-step semantics relates *prestates* of statement executions to *poststates*
- Often called *natural semantics*, because it abstracts from intermediate states

Overview:

- Basic concepts of big-step semantics
- Formalization of big-step semantics in Isabelle/HOL

Subsection 7.2.1

Basic concepts of big-step semantics

Transitions in Natural Semantics

- Two types of configurations for operational semantics
  1. $\langle s, \sigma \rangle$, which represents that the statement s is to be executed in state σ
  2. σ, which represents a terminal state
- The transition relation $\rightarrow$ describes how executions take place
  - Typical transition: $\langle s, \sigma \rangle \rightarrow \sigma'$
  - Example: $\langle \texttt{skip}, \sigma \rangle \rightarrow \sigma$

$$\Gamma = \{\langle s, \sigma \rangle \mid s \in \text{Stm}, \sigma \in \text{State}\} \cup \text{State}$$
$$T = \text{State}$$
$$\rightarrow \subseteq \{\langle s, \sigma \rangle \mid s \in \text{Stm}, \sigma \in \text{State}\} \times \text{State}$$

Rules

- Transition relation is specified by rules

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi} \quad \text{if } \textit{Condition}$$

  where $\varphi_1, \ldots, \varphi_n$ and ψ are transitions

- Meaning of the rule

If *Condition* and $\varphi_1, \ldots, \varphi_n$ then ψ

- Terminology
  - $\varphi_1, \ldots, \varphi_n$ are called **premises**
  - ψ is called **conclusion**
  - A rule without premises is called **axiom**

Notation

- Updating States: $\sigma[y \mapsto v]$ is the function that
  - overrides the association of y in σ by $y \mapsto v$ or
  - adds the new association $y \mapsto v$ to σ

$$(\sigma[y \mapsto v])(x) = \begin{cases} v & \text{if } x = y \\ \sigma(x) & \text{if } x \neq y \end{cases}$$

Natural Semantics of IMP

- `skip` does not modify the state

$$\overline{\langle \texttt{skip}, \sigma \rangle \rightarrow \sigma}$$

- x`:=`e assigns the value of e to variable e

$$\overline{\langle x\texttt{:=}e, \sigma \rangle \rightarrow \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]}$$

- Sequential composition $s_1\texttt{;}s_2$
  - First, s_1 is executed in state σ, leading to σ'
  - Then s_2 is executed in state σ'

$$\frac{\langle s_1, \sigma \rangle \rightarrow \sigma', \langle s_2, \sigma' \rangle \rightarrow \sigma''}{\langle s_1\texttt{;}s_2, \sigma \rangle \rightarrow \sigma''}$$

Natural Semantics of IMP (cont'd)

- Conditional statement $\texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}$
  - If b holds, s_1 is executed
  - If b does not hold, s_2 is executed

$$\frac{\langle s_1, \sigma\rangle \to \sigma'}{\langle \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}, \sigma\rangle \to \sigma'} \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

$$\frac{\langle s_2, \sigma\rangle \to \sigma'}{\langle \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}, \sigma\rangle \to \sigma'} \quad \text{if} \mathcal{B}[\![b]\!]\sigma = ff$$

Natural Semantics of IMP (cont'd)

- Loop statement `while` b `do` s `end`
  - If b holds, s is executed once, leading to state σ'
  - Then the whole while-statement is executed again σ'

$$\frac{\langle s, \sigma\rangle \to \sigma', \langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma'\rangle \to \sigma''}{\langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \to \sigma''} \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

  - If b does not hold, the while-statement does not modify the state

$$\frac{}{\langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \to \sigma} \quad \text{if } \mathcal{B}[\![b]\!]\sigma = ff$$

Rule Instantiations

- Rules are actually **rule schemes**
  - Meta-variables stand for arbitrary variables, expressions, statements, states, etc.
  - To apply rules, they have to be **instantiated** by selecting particular variables, expressions, statements, states, etc.
- Assignment rule **scheme**

$$\langle x := e, \sigma \rangle \rightarrow \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$$

- Assignment rule **instance**

$$\langle \mathsf{v} := \mathsf{v}+1, \{\mathsf{v} \mapsto 3\} \rangle \rightarrow \{\mathsf{v} \mapsto 4\}$$

Derivations: Example

- What is the final state if statement

```
z:=x; x:=y; y:=z
```

is executed in state $\{x \mapsto 5, y \mapsto 7, z \mapsto 0\}$ (abbreviated by $[5, 7, 0]$)?

$$\frac{\dfrac{\langle z{:=}x, [5,7,0]\rangle \to [5,7,5], \langle x{:=}y, [5,7,5]\rangle \to [7,7,5]}{\langle z{:=}x;\ x{:=}y, [5,7,0]\rangle \to [7,7,5]}, \quad \langle y{:=}z, [7,7,5]\rangle \to [7,5,5]}{\langle z{:=}x;\ x{:=}y;\ y{:=}z, [5,7,0]\rangle \to [7,5,5]}$$

Derivation Trees

- Rule instances can be combined to derive a transition $\langle s, \sigma \rangle \rightarrow \sigma'$
- The result is a **derivation tree**
  - The root is the transition $\langle s, \sigma \rangle \rightarrow \sigma'$
  - The leaves are axiom instances
  - The internal nodes are conclusions of rule instances and have the corresponding premises as immediate children
- The conditions of all instantiated rules must be satisfied
- There can be several derivations for one transition (non-deterministic semantics)

Termination

- The execution of a statement s in state σ
  - **terminates** iff there is a state σ' such that $\langle s, \sigma \rangle \rightarrow \sigma'$
  - **loops** iff there is no state σ' such that $\langle s, \sigma \rangle \rightarrow \sigma'$

- A statement s
  - **always terminates** if the execution in a state σ terminates for all choices of σ
  - **always loops** if the execution in a state σ loops for all choices of σ

Semantic Equivalence

- Definition

  Two statements s_1 and s_2 are **semantically equivalent** (denoted by $s_1 \equiv s_2$) if the following property holds for all states σ, σ':

  $$\langle s_1, \sigma \rangle \rightarrow \sigma' \Leftrightarrow \langle s_2, \sigma \rangle \rightarrow \sigma'$$

- Example

  `while` b `do` s `end` $\equiv$
  `if` b `then` s`;` `while` b `do` s `end`

Subsection 7.2.2

Formalization of big-step semantics

Big-step semantics in Isabelle/HOL

Approach

- Formalize the abstract syntax of the language by a recursive datatype
- Formalize the big-step semantics as an inductive predicate

Arithmetic expressions

```
datatype var = V nat   (".v_"   90)

datatype aexp = Plus  aexp aexp     (infixr ".+."   30)
              | Minus aexp aexp     (infixr ".-."   30)
              | Mult  aexp aexp     (infixr ".*."   50)
              | Div   aexp aexp     (infixr "./."   50)
              | Mod   aexp aexp     (infixr ".%."   50)
              | Var   var           ("'_" 90)
              | Const int           ("'_" 90)
```

Boolean expressions

```
datatype bexp = Or   bexp bexp      (infixr ".|."  20)
              | And  bexp bexp      (infixr ".&."  20)
              | Not  bexp           (".!_" 80)
              | Eq   aexp aexp      (infixr ".=."  10)
              | Less aexp aexp      (infixr ".<."   10)
              | Leq  aexp aexp      (infixr ".<=."  10)
```

commands/statements

```
datatype com  =
    Skip                        ("SKIP")
  | Assign var aexp             ("_:==_" [60, 60] 20)
  | Semi   com com              ("_; _"  [60, 60] 10)
  | Cond   bexp com com         ("IF _ THEN _ ELSE _ END"  60)
  | While bexp com              ("WHILE _ DO _ END"  60)
```

Evaluation of arithmetic expressions

```
type_synonym state = "var ⇒ int"

primrec  aeval :: "aexp ⇒ state ⇒ int" where
  "aeval (Plus  la ra) s = ((aeval la s) + (aeval ra s))"
| "aeval (Minus la ra) s = ((aeval la s) - (aeval ra s))"
| "aeval (Mult  la ra) s = ((aeval la s) * (aeval ra s))"
| "aeval (Div   la ra) s = ((aeval la s)div(aeval ra s))"
| "aeval (Mod   la ra) s = ((aeval la s)mod(aeval ra s))"
| "aeval (Var   v) s     = (s v)"
| "aeval (Const i) s     = i"
```

Evaluation of boolean expressions

```
primrec  beval :: "bexp ⇒ state ⇒ bool" where
  "beval (Or   lb rb) s = ((beval lb s) ∨ (beval rb s))"
| "beval (And  lb rb) s = ((beval lb s) ∧ (beval rb s))"
| "beval (Not  be) s    = (¬ (beval be s))"
| "beval (Eq   la ra) s = ((aeval la s) = (aeval ra s))"
| "beval (Less la ra) s = ((aeval la s) < (aeval ra s))"
| "beval (Leq  la ra) s = ((aeval la s) ≤ (aeval ra s))"
```

Operational semantics

```
inductive   exec :: "state ⇒ com ⇒ state ⇒ bool"
                    ("_/ -_→/ _" [50,0,50] 50) where
  Skip:   "s -SKIP→ s"
| Assign: "s -(v:== ae)→ s(v:= aeval ae s)"

| Semi:   "⟦ s0 -c1→ s1; s1 -c2→ s2 ⟧
          ⟹ s0 -c1;c2→ s2"

| IfT:    "⟦ beval b s  ; s -c1→ t ⟧
          ⟹ s -IF b THEN c1 ELSE c2 END→ t"
| IfF:    "⟦ ¬(beval b s); s -c2→ t ⟧
          ⟹ s -IF b THEN c1 ELSE c2 END→ t"

| WhileF: "¬(beval b s) ⟹ s -WHILE b DO c END→ s"
| WhileT: "⟦ beval b s; s-c→t; t -WHILE b DO c END→ u ⟧
          ⟹ s -WHILE b DO c END→ u"
```

Some properties

```
lemma [iff]: "(s -c;d→ u) = (∃t. s -c→ t ∧ t -d→ u)"

lemma [iff]: "(s -IF be THEN c ELSE d END→ t) =
              (s -if beval be s then c else d → t)"

lemma unfold_while:
 "(s -WHILE b DO c END→ u) =
  (s -IF b THEN c; WHILE b DO c  END ELSE SKIP END→ u)"

lemma while_rule:
 "⟦ s -WHILE b DO c END→ t; P s;
   ∀s s'. P s ∧ (beval b s) ∧ s -c→ s' ⟶ P s'⟧
  ⟹ P t ∧ ¬(beval b t)"
```

Formalization of semantics and verification

Remarks

- Inductive definition of the “semantics judgement” leads to a semantic predicate satisfying the least fixpoint of the semantical rules
- The operational semantics can be directly used for program verification (Why do we need a programming logic? (cf. Chapter 8))

Section 7.3

Small-step semantics

Overview

Subsection 7.3.1

Small-step semantics of IMP

Structural Operational Semantics

- The emphasis is on the **individual steps** of the execution
  - Execution of assignments
  - Execution of tests
- Describing small steps of the execution allows one to express the **order of execution** of individual steps
  - Interleaving computations
  - Evaluation order for expressions (not shown in the course)
- Describing always the **next small step** allows one to express **properties of looping programs**

Transitions in SOS

- The configurations are the same as for natural semantics
- The transition relation $\rightarrow_1$ can have two forms
- $\langle s, \sigma \rangle \rightarrow_1 \langle s', \sigma' \rangle$: the execution of s from σ is **not completed** and the remaining computation is expressed by the intermediate configuration $\langle s', \sigma' \rangle$
- $\langle s, \sigma \rangle \rightarrow_1 \sigma'$: the execution of s from σ **has terminated** and the final state is σ'
- A transition $\langle s, \sigma \rangle \rightarrow_1 \gamma$ describes the **first step** of the execution of s from σ

Transition System

$$\Gamma = \{\langle s, \sigma\rangle \mid s \in \text{Stm}, \sigma \in \text{State}\} \cup \text{State}$$
$$T = \text{State}$$
$$\rightarrow_1 \subseteq \{\langle s, \sigma\rangle \mid s \in \text{Stm}, \sigma \in \text{State}\} \times \Gamma$$

- We say that $\langle s, \sigma\rangle$ is **stuck** if there is no γ such that $\langle s, \sigma\rangle \rightarrow_1 \gamma$

SOS of IMP

- `skip` does not modify the state

$$\langle \texttt{skip}, \sigma \rangle \to_1 \sigma$$

- x`:=`e assigns the value of e to variable x

$$\langle x \texttt{:=} e, \sigma \rangle \to_1 \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$$

- `skip` and assignment require only one step
- Rules are analogous to natural semantics

$$\langle \texttt{skip}, \sigma \rangle \to \sigma$$

$$\langle x \texttt{:=} e, \sigma \rangle \to \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$$

SOS of IMP: Sequential Composition

- Sequential composition $s_1 ; s_2$
- First step of executing $s_1 ; s_2$ is the first step of executing s_1
- s_1 is executed in one step

$$\frac{\langle s_1, \sigma \rangle \rightarrow_1 \sigma'}{\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \langle s_2, \sigma' \rangle}$$

- s_1 is executed in several steps

$$\frac{\langle s_1, \sigma \rangle \rightarrow_1 \langle s_1', \sigma' \rangle}{\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \langle s_1' ; s_2, \sigma' \rangle}$$

SOS of IMP: Conditional Statement

- The first step of executing `if` b `then` s_1 `else` s_2 `end` is to determine the outcome of the test and thereby which branch to select

$$\langle \texttt{if } b \texttt{ then } s_1 \texttt{ else } s_2 \texttt{ end}, \sigma\rangle \to_1 \langle s_1, \sigma\rangle \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

$$\langle \texttt{if } b \texttt{ then } s_1 \texttt{ else } s_2 \texttt{ end}, \sigma\rangle \to_1 \langle s_2, \sigma\rangle \quad \text{if } \mathcal{B}[\![b]\!]\sigma = ff$$

Alternative for Conditional Statement

- The first step of executing $\texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}$ is the first step of the branch determined by the outcome of the test

$$\frac{\langle s_1, \sigma\rangle \to_1 \sigma'}{\langle \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}, \sigma\rangle \to_1 \sigma'} \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

$$\frac{\langle s_1, \sigma\rangle \to_1 \langle s_1', \sigma'\rangle}{\langle \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}, \sigma\rangle \to_1 \langle s_1', \sigma'\rangle} \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

and two similar rules for $\mathcal{B}[\![b]\!]\sigma = ff$

- Alternatives are equivalent for IMP
- Choice is important for languages with parallel execution

SOS of IMP: Loop Statement

- The first step is to unrole the loop

$$\langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \rightarrow_1 \langle \texttt{if}\ b\ \texttt{then}\ s\texttt{;while}\ b\ \texttt{do}\ s\ \texttt{end}\ \texttt{else}\ \texttt{skip}\ \texttt{end}, \sigma\rangle$$

- Recall that $\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}$ and $\texttt{if}\ b\ \texttt{then}\ s\texttt{;while}\ b\ \texttt{do}\ s\ \texttt{end}\ \texttt{else}\ \texttt{skip}\ \texttt{end}$ are semantically equivalent in the natural semantics

Alternatives for Loop Statement

- The first step is to decide the outcome of the test and thereby whether to unrole the body of the loop or to terminate

$$\langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \to_1 \langle s\texttt{;while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \quad \text{if } \mathcal{B}[\![b]\!]\sigma = tt$$

$$\langle \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}, \sigma\rangle \to_1 \sigma \quad \text{if } \mathcal{B}[\![b]\!]\sigma = ff$$

- Or combine with the alternative semantics of the conditional statement
- Alternatives are equivalent for IMP

Derivation Sequences

- A **derivation sequence** of a statement s starting in state σ is a sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$, where
  - $\gamma_0 = \langle s, \sigma \rangle$
  - $\gamma_i \rightarrow_1 \gamma_{i+1}$ for $0 \leq i$
- A derivation sequence is either **finite** or **infinite**
  - Finite derivation sequences end with a configuration that is either a terminal configuration or a stuck configuration
- Notation
  - $\gamma_0 \rightarrow_1^i \gamma_i$ indicates that there are i steps in the execution from γ_0 to γ_i
  - $\gamma_0 \rightarrow_1^* \gamma_i$ indicates that there is a **finite number of steps** in the execution from γ_0 to γ_i
  - $\gamma_0 \rightarrow_1^i \gamma_i$ and $\gamma_0 \rightarrow_1^* \gamma_i$ need **not** be derivation sequences

Derivation Sequences: Example

- What is the final state if statement

```
z:=x; x:=y; y:=z
```

is executed in state $\{\mathsf{x} \mapsto 5, \mathsf{y} \mapsto 7, \mathsf{z} \mapsto 0\}$?

$$\langle \texttt{z:=x; x:=y; y:=z}, \{\mathsf{x} \mapsto 5, \mathsf{y} \mapsto 7, \mathsf{z} \mapsto 0\}\rangle$$
$$\rightarrow_1 \langle \texttt{x:=y; y:=z}, \{\mathsf{x} \mapsto 5, \mathsf{y} \mapsto 7, \mathsf{z} \mapsto 5\}\rangle$$
$$\rightarrow_1 \langle \texttt{y:=z}, \{\mathsf{x} \mapsto 7, \mathsf{y} \mapsto 7, \mathsf{z} \mapsto 5\}\rangle$$
$$\rightarrow_1 \{\mathsf{x} \mapsto 7, \mathsf{y} \mapsto 5, \mathsf{z} \mapsto 5\}$$

Derivation Trees

- Derivation trees explain why transitions take place
- For the first step

$$\langle \texttt{z:=x; x:=y; y:=z}, \sigma\rangle \rightarrow_1 \langle \texttt{x:=y; y:=z}, \sigma[\texttt{z} \mapsto 5]\rangle$$

the derivation tree is

$$\dfrac{\dfrac{\langle \texttt{z:=x}, \sigma\rangle \rightarrow_1 \sigma[\texttt{z} \mapsto 5]}{\langle \texttt{z:=x; x:=y}, \sigma\rangle \rightarrow_1 \langle \texttt{x:=y}, \sigma[\texttt{z} \mapsto 5]\rangle}}{\langle \texttt{z:=x; x:=y; y:=z}, \sigma\rangle \rightarrow_1 \langle \texttt{x:=y; y:=z}, \sigma[\texttt{z} \mapsto 5]\rangle}$$

- `z:=x; ( x:=y; y:=z )` would lead to a simpler tree with only one rule application

Derivation Sequences and Trees

- Natural (big-step) semantics
  - The execution of a statement (sequence) is described by one big transition
  - The big transition can be seen as trivial derivation sequence with exactly one transition
  - The derivation tree explains why this transition takes place
- Structural operational (small-step) semantics
  - The execution of a statement (sequence) is described by one or more transitions
  - Derivation sequences are important
  - Derivation trees justify each individual step in a derivation sequence

Termination

- The execution of a statement s in state σ
  - **terminates** iff there is a finite derivation sequence starting with $\langle s, \sigma \rangle$
  - **loops** iff there is an infinite derivation sequence starting with $\langle s, \sigma \rangle$

- The execution of a statement s in state σ
  - **terminates successfully** if $\langle s, \sigma \rangle \rightarrow_1^* \sigma'$
  - In IMP, an execution terminates successfully iff it terminates (no stuck configurations)

Subsection 7.3.2

Proving properties of the semantics

Induction on Derivations

Induction on the length of derivation sequences

1. **Induction base**: Prove that the property holds for all derivation sequences of length 0
2. **Induction step**: Prove that the property holds for all other derivation sequences:
   - **Induction hypothesis**: Assume that the property holds for all derivation sequences of length at most k
   - Prove that it also holds for derivation sequences of length $k + 1$

Induction on the length of derivation sequences is an application of strong mathematical induction.

Using Induction on Derivations

- The induction step is often done by inspecting either
  - the structure of the syntactic element or
  - the derivation tree validating the first transition of the derivation sequence

- Lemma

$$\langle s_1 ; s_2, \sigma\rangle \rightarrow_1^k \sigma'' \Rightarrow$$
$$\exists \sigma', k_1, k_2 : \ \langle s_1, \sigma\rangle \rightarrow_1^{k_1} \sigma' \wedge \langle s_2, \sigma'\rangle \rightarrow_1^{k_2} \sigma'' \wedge$$
$$k_1 + k_2 = k$$

Proof

- Proof by induction on k, that is, by induction on the length of the derivation sequence for $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1^k \sigma''$
- Induction base: $k = 0$: There is no derivation sequence of length 0 for $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1^k \sigma''$
- Induction step
  - We assume that the lemma holds for $k \leq m$
  - We prove that the lemma holds for $m + 1$
  - The derivation sequence $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1^{m+1} \sigma''$ can be written as $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1 \gamma \rightarrow_1^m \sigma''$ for some configuration γ

Induction Step

- $\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \gamma \rightarrow_1^m \sigma''$
- Consider the two rules that could lead to the transition $\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \gamma$
- Case 1

$$\frac{\langle s_1, \sigma \rangle \rightarrow_1 \sigma'}{\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \langle s_2, \sigma' \rangle}$$

- Case 2

$$\frac{\langle s_1, \sigma \rangle \rightarrow_1 \langle s_1', \sigma' \rangle}{\langle s_1 ; s_2, \sigma \rangle \rightarrow_1 \langle s_1' ; s_2, \sigma' \rangle}$$

Induction Step: Case 1

- From
  $\langle s_1 ; s_2, \sigma\rangle \to_1 \gamma \to_1^m \sigma''$ and $\langle s_1 ; s_2, \sigma\rangle \to_1 \langle s_2, \sigma'\rangle$
  we conclude $\langle s_2, \sigma'\rangle \to_1^m \sigma''$

- The required result follows by choosing $k_1 = 1$ and $k_2 = m$

Induction Step: Case 2

- From
  $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1 \gamma \rightarrow_1^m \sigma''$ and $\langle s_1 ; s_2, \sigma\rangle \rightarrow_1 \langle s_1' ; s_2, \sigma'\rangle$
  we conclude $\langle s_1' ; s_2, \sigma'\rangle \rightarrow_1^m \sigma''$
- By applying the induction hypothesis, we get
  $\exists \sigma_0, l_1, l_2 : \langle s_1', \sigma'\rangle \rightarrow_1^{l_1} \sigma_0 \wedge \langle s_2, \sigma_0\rangle \rightarrow_1^{l_2} \sigma'' \wedge l_1 + l_2 = m$
- From
  $\langle s_1, \sigma\rangle \rightarrow_1 \langle s_1', \sigma'\rangle$ and $\langle s_1', \sigma'\rangle \rightarrow_1^{l_1} \sigma_0$
  we get $\langle s_1, \sigma\rangle \rightarrow_1^{l_1+1} \sigma_0$
- By
  $\langle s_2, \sigma_0\rangle \rightarrow_1^{l_2} \sigma''$ and $(l_1 + 1) + l_2 = m + 1$
  we have proved the required result

Semantic Equivalence

Two statements s_1 and s_2 are **semantically equivalent** if for all states σ:

- $\langle s_1, \sigma \rangle \rightarrow_1^* \gamma$ iff $\langle s_2, \sigma \rangle \rightarrow_1^* \gamma$, whenever γ is a configuration that is either stuck or terminal, and
- there is an infinite derivation sequence starting in $\langle s_1, \sigma \rangle$ iff there is one starting in $\langle s_2, \sigma \rangle$

Note: In the first case, the length of the two derivation sequences may be different

Determinism

Lemma: The structural operational semantics of IMP is deterministic. That is, for all s,σ,γ, and γ' we have that

$$\langle s, \sigma\rangle \to_1 \gamma \wedge \langle s, \sigma\rangle \to_1 \gamma' \Rightarrow \gamma = \gamma'$$

- The proof runs by induction on the shape of the derivation tree for the transition $\langle s, \sigma\rangle \to_1 \gamma$

Corollary: There is exactly one derivation sequence starting in configuration $\langle s, \sigma\rangle$

- The proof runs by induction on the length of the derivation sequence

Subsection 7.3.3

Extensions of IMP

Extensions of IMP

- Local variable declarations
- Statement “abort”
- Non-determinism
- Parallelism

Local Variable Declarations

- Local variable declaration `var` x`:=`e `in` s `end`
- The small steps are
  1. Assign e to x
  2. Execute s
  3. Restore the initial value of x
     (necessary if x exists in the enclosing scope)

- Problem: There is no history of states that could be used to restore the value of x
- Idea: Represent states as execution stacks

Modelling Execution Stacks

- We model execution stacks by providing a mapping $\text{Var} \to \text{Val}$ for each scope

$$\text{State} : \textit{stack of}(\text{Var} \to \text{Val})$$

- Assignment and lookup have to determine the highest stack element in which a variable is defined
- Example: $\sigma(x) = 3$

$z \mapsto 4$
$x \mapsto 3$
$x \mapsto 1, y \mapsto 2$

SOS for Variable Declarations

- The small steps are
  1. Create new scope and assign e to x in this scope
  2. Execute s
  3. Restore the initial value of x using a `return` statement

$$\langle \texttt{var } x \texttt{:=} e \texttt{ in } s \texttt{ end}, \sigma\rangle \rightarrow_1 \langle s\texttt{;return}, \mathit{push}(\{x \mapsto \mathcal{A}[\![e]\!]\sigma\}, \sigma)\rangle$$

$$\langle \texttt{return}, \sigma\rangle \rightarrow_1 \mathit{pop}(\sigma)$$

- Similar techniques can be used for procedure calls

Abortion

- Statement `abort` stops the execution of the complete program
- Abortion is modeled by ensuring that the configurations $\langle \texttt{abort}, \sigma \rangle$ are **stuck**
- There is no additional rule for `abort` in the structural operational semantics
- `abort` and `skip` are not semantically equivalent
  - $\langle \texttt{abort}, \sigma \rangle$ is the only derivation sequence for `abort` starting is s
  - $\langle \texttt{skip}, \sigma \rangle \rightarrow_1 \sigma$ is the only derivation sequence for `skip` starting is s

Abortion: Observations

- `abort` and `while true do skip end` are not semantically equivalent:

$\langle$`while true do skip end`$, \sigma\rangle \rightarrow_1$

$\langle$`if true then skip;while true do skip end end`$, \sigma\rangle \rightarrow_1$

$\langle$`skip;while true do skip end`$\rangle \rightarrow_1$

$\langle$`while true do skip end`$, \sigma\rangle$

- In a structural operational semantics,
  - looping is reflected by infinite derivation sequences
  - abnormal termination by finite derivation sequences ending in a stuck configuration

Non-determinism

- For the statement $s_1 \,[]\, s_2$ either s_1 or s_2 is non-deterministically chosen to be executed
- The statement

```
x:=1[]x:=2; x:=x+2
```

  could result in a state in which x has the value 1 or 4
- Rules

$$\langle s_1 \,[]\, s_2, \sigma\rangle \to_1 \langle s_1, \sigma\rangle \qquad \langle s_1 \,[]\, s_2, \sigma\rangle \to_1 \langle s_2, \sigma\rangle$$

Non-determinism: Observations

- There are two derivation sequences
  - $\langle \texttt{x:=1[]x:=2; x:=x+2}, \sigma\rangle \rightarrow_1^* \sigma[\texttt{x} \mapsto 1]$
  - $\langle \texttt{x:=1[]x:=2; x:=x+2}, \sigma\rangle \rightarrow_1^* \sigma[\texttt{x} \mapsto 4]$
- There are also two derivation sequences for $\langle \texttt{while true do skip end[]x:=2; x:=x+2}, \sigma\rangle$
  - an finite derivation sequence leading to $\sigma[\texttt{x} \mapsto 4]$
  - an infinite derivation sequence
- A structural operational semantics can choose the ”wrong” branch of a non-deterministic choice
- In a structural operational semantics **non-determinism does not suppress looping**

Parallelism

- For the statement s_1 par s_2 both statements s_1 and s_2 are executed, but execution can be **interleaved**

$$\frac{\langle s_1, \sigma\rangle \to_1 \langle s_1', \sigma'\rangle}{\langle s_1 \text{ par } s_2, \sigma\rangle \to_1 \langle s_1' \text{ par } s_2, \sigma'\rangle}$$

$$\frac{\langle s_1, \sigma\rangle \to_1 \sigma'}{\langle s_1 \text{ par } s_2, \sigma\rangle \to_1 \langle s_2, \sigma'\rangle}$$

$$\frac{\langle s_2, \sigma\rangle \to_1 \langle s_2', \sigma'\rangle}{\langle s_1 \text{ par } s_2, \sigma\rangle \to_1 \langle s_1 \text{ par } s_2', \sigma'\rangle}$$

$$\frac{\langle s_2, \sigma\rangle \to_1 \sigma'}{\langle s_1 \text{ par } s_2, \sigma\rangle \to_1 \langle s_1, \sigma'\rangle}$$

Example: Interleaving

- The statement

  ```
  x:=1 par x:=2; x:=x+2
  ```

  could result in a state in which x has the value 4, 1, or 3
  - Execute x:=1, then x:=2, and then x:=x+2
  - Execute x:=2, then x:=x+2, and then x:=1
  - Execute x:=2, then x:=1, and then x:=x+2

- In a structural operational semantics we can easily express interleaving of computations

Example: Derivation Sequences

$$\langle \texttt{x:=1 par x:=2; x:=x+2}, \sigma\rangle \to_1 \langle \texttt{x:=2; x:=x+2}, \sigma[x \mapsto 1]\rangle$$
$$\to_1 \langle \texttt{x:=x+2}, \sigma[x \mapsto 2]\rangle$$
$$\to_1 \sigma[x \mapsto 4]$$

$$\langle \texttt{x:=1 par x:=2; x:=x+2}, \sigma\rangle \to_1 \langle \texttt{x:=1 par x:=x+2}, \sigma[x \mapsto 2]\rangle$$
$$\to_1 \langle \texttt{x:=1}, \sigma[x \mapsto 4]\rangle$$
$$\to_1 \sigma[x \mapsto 1]$$

$$\langle \texttt{x:=1 par x:=2; x:=x+2}, \sigma\rangle \to_1 \langle \texttt{x:=1 par x:=x+2}, \sigma[x \mapsto 2]\rangle$$
$$\to_1 \langle \texttt{x:=x+2}, \sigma[x \mapsto 1]\rangle$$
$$\to_1 \sigma[x \mapsto 3]$$

Comparison: Summary

Natural Semantics

- Local variable declarations and procedures can be modeled easily
- No distinction between abortion and looping
- Non-determinism suppresses looping (if possible)
- Parallelism cannot be modeled

Structural Operational Semantics

- Local variable declarations and procedures require modeling the execution stack
- Distinction between abortion and looping
- Non-determinism does not suppress looping
- Parallelism can be modeled

Section 7.4

Denotational semantics

Motivation

Goals of a semantics definition

- Semantics defines *observational* behavior
- Semantics defines an equivalence relation on **programs**:
  When are two programs considered to be equal?
- Semantics should also provide semantics of **program parts**:
  E.g.: When can a Java class be used for another class?
- Semantics should be defined compositional

Observations

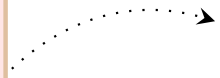
Operational semantics:

- often not sufficiently abstract and non-compositional
- often unclear how to handle program parts

```
package cells;

public interface Val{}

public class Cell {
 private Val vatt;
 public void set(Val v)
 {
   vatt = v;
 }
 public Val get() {
   return vatt;
 }
}
```



```
package cells;

public interface Val {}

public class Cell {
 private Val v1, v2;
 private boolean f;
 public void set(Val v) {
  f = !f ;
  if (f) v1 = v;
  else v2 = v;
 }
 public Val get() {
  return f ? v1 : v2;
 }
 public Val getPrevious() {
  return f ? v2 : v1;
}}
```

Approach

- Denotational semantics describes the **effect** of a computation

- A semantic function is defined for each syntactic construct
  - maps syntactic construct to a mathematical object, often a function
  - the mathematical object describes the effect of executing the syntactic construct

Compositionality

- In denotational semantics, semantic functions are defined **compositionally**
- There is a semantic clause for each of the basis elements of the syntactic category
- For each method of constructing a composite element (in the syntactic category) there is a semantic clause defined in terms of the **semantic function applied to the immediate constituents** of the composite element

Examples

- The semantic functions $\mathcal{A}$: Aexp $\to$ State $\to$ Val and $\mathcal{B}$: Bexp $\to$ State $\to$ Bool are denotational definitions

$$
\begin{array}{lll}
\mathcal{A}[\![x]\!]\sigma & = \sigma(x) & \\
\mathcal{A}[\![i]\!]\sigma & = i & \text{for } i \in \mathbb{Z} \\
\mathcal{A}[\![e_1\ op\ e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma & \text{for } op \in \text{Op}
\end{array}
$$

$$
\mathcal{B}[\![e_1\ op\ e_2]\!]\sigma = \begin{cases} tt & \text{if } \mathcal{A}[\![e_1]\!]\sigma\ \overline{op}\ \mathcal{A}[\![e_2]\!]\sigma \\ ff & \text{otherwise} \end{cases}
$$

Counterexamples

- The semantic functions $\mathcal{S}_{NS}$ and $\mathcal{S}_{SOS}$ are not denotational definitions because they are not defined compositionally

$$\mathcal{S}_{NS} : \text{Stm} \to (\text{State} \hookrightarrow \text{State})$$

$$\mathcal{S}_{NS}[\![s]\!]\sigma = \begin{cases} \sigma' & \text{if } \langle s, \sigma\rangle \to \sigma' \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\mathcal{S}_{SOS} : \text{Stm} \to (\text{State} \hookrightarrow \text{State})$$

$$\mathcal{S}_{SOS}[\![s]\!]\sigma = \begin{cases} \sigma' & \text{if } \langle s, \sigma\rangle \to_1^* \sigma' \\ \text{undefined} & \text{otherwise} \end{cases}$$

Semantic Functions

- The effect of executing a statement is described by the partial function $\mathcal{S}_{DS}$

$$\mathcal{S}_{DS} : \text{Stm} \to (\text{State} \hookrightarrow \text{State})$$

- Partiality is needed to model non-termination
- The effects of evaluating expressions is defined by the functions $\mathcal{A}$ and $\mathcal{B}$

Direct Style Semantics of IMP

- `skip` does not modify the state

$$\mathcal{S}_{DS}[\![\texttt{skip}]\!] = id$$

$$id : \text{State} \to \text{State}$$

$$id(\sigma) = \sigma$$

- x `:=` e assigns the value of e to variable x

$$\mathcal{S}_{DS}[\![x\texttt{:=}e]\!]\sigma = \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$$

Direct Style Semantics of IMP (cont'd)

- Sequential composition $s_1 ; s_2$

$$\mathcal{S}_{DS}[\![s_1 ; s_2]\!] = \mathcal{S}_{DS}[\![s_2]\!] \circ \mathcal{S}_{DS}[\![s_1]\!]$$

- Function composition $\circ$ is defined in a **strict** way
  - If one of the functions is undefined on the given argument then the composition is undefined

$$(f \circ g)\sigma = \begin{cases} f(g(\sigma)) & \text{if } g(\sigma) \neq \text{undefined} \\ & \text{and } f(g(\sigma)) \neq \text{undefined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Direct Style Semantics of IMP (cont'd)

- Conditional statement $\texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}$

$$\mathcal{S}_{DS}[\![\texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}]\!] = cond(\mathcal{B}[\![b]\!], \mathcal{S}_{DS}[\![s_1]\!], \mathcal{S}_{DS}[\![s_2]\!])$$

- The function $cond$
  - takes the semantic functions for the condition and the two statements
  - when supplied with a state selects the second or third argument depending on the first

$$cond : (\text{State} \to \text{Bool}) \times (\text{State} \hookrightarrow \text{State}) \times (\text{State} \hookrightarrow \text{State}) \to (\text{State} \hookrightarrow \text{State})$$

Definition of $cond$

$$cond : \big(\text{State} \to \text{Bool}\big) \times \big(\text{State} \hookrightarrow \text{State}\big) \times \big(\text{State} \hookrightarrow \text{State}\big) \to \big(\text{State} \hookrightarrow \text{State}\big)$$

$$cond(b, f, g)\sigma = \begin{cases} f(\sigma) & \text{if } b(\sigma) = tt \\ & \text{and } f(\sigma) \neq \text{undefined} \\ g(\sigma) & \text{if } b(\sigma) = ff \\ & \text{and } g(\sigma) \neq \text{undefined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Semantics of Loop: Observations

- Defining the semantics of `while` is difficult
- The semantics of `while` b `do` s `end` must be equal to `if` b `then` s`;while` b `do` s `end else skip end`
- This requirement yields:

$$\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] = cond(\mathcal{B}[\![b]\!], \mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] \circ \mathcal{S}_{DS}[\![s]\!], id)$$

- We cannot use this equation as a definition because it is not compositional

Functionals and Fixed Points

$$\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] = cond(\mathcal{B}[\![b]\!], \mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] \circ \mathcal{S}_{DS}[\![s]\!], id)$$

- The above equation has the form $g = F(g)$
  - $g = \mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!]$
  - $F(g) = cond(\mathcal{B}[\![b]\!], g \circ \mathcal{S}_{DS}[\![s]\!], id)$
- F is a **functional** (a function from functions to functions)
- $\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!]$ is a **fixed point** of the functional F

Fixed Points: Examples

- x is a fixed point of function f if $f(x) = x$ holds

- Consider a function $f : \mathbb{N} \to \mathbb{N}$
  - $f(x) = x + 1$ does not have a fixed point
  - $f(x) = 0$ has exactly one fixed point, 0
  - $f(x) = x^2$ has two fixed points, 0 and 1
  - $f(x) = x$ has an infinite number of fixed points

Direct Style Semantics of IMP: Loops

- Loop statement `while` b `do` s `end`

$$\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] = FIX\ F$$
$$\text{where } F(g) = cond(\mathcal{B}[\![b]\!], g \circ \mathcal{S}_{DS}[\![s]\!], id)$$

- We write $FIX\ F$ to denote the fixed point of the functional F:

$$FIX\ :\ ((\text{State} \hookrightarrow \text{State}) \rightarrow (\text{State} \hookrightarrow \text{State})) \rightarrow (\text{State} \hookrightarrow \text{State})$$

- This defintion of $\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!]$ is compositional

Example

- Consider the statement

```
while x # 0 do skip end
```

- The functional for this loop is defined by

$$
\begin{aligned}
F'(g)\sigma &= cond(\mathcal{B}[\![\texttt{x\#0}]\!], g \circ \mathcal{S}_{DS}[\![\texttt{skip}]\!], id)\sigma \\
&= cond(\mathcal{B}[\![\texttt{x\#0}]\!], g \circ id, id)\sigma \\
&= cond(\mathcal{B}[\![\texttt{x\#0}]\!], g, id)\sigma \\
&= \begin{cases} g(\sigma) & \text{if } \sigma(x) \neq 0 \\ \sigma & \text{if } \sigma(x) = 0 \end{cases}
\end{aligned}
$$

Example (cont'd)

- The function

$$g_1(\sigma) = \begin{cases} \text{undefined} & \text{if } \sigma(x) \neq 0 \\ \sigma & \text{if } \sigma(x) = 0 \end{cases}$$

  is a fixed point of F'

- The function $g_2(\sigma) = \text{undefined}$ is not a fixed point for F'

Well-Definedness

$$\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!] = FIX\ F$$
$$\text{where } F(g) = cond(\mathcal{B}[\![b]\!], g \circ \mathcal{S}_{DS}[\![s]\!], id)$$

- The function $\mathcal{S}_{DS}[\![\texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}]\!]$ is well-defined if $FIXF$ defines a **unique fixed point** for the functional F
  - There are functionals that have more than one fixed point
  - There are functionals that have no fixed point at all

Examples

- F' from the previous example has more than one fixed point

$$F'(g)\sigma = \begin{cases} g(\sigma) & \text{if } \sigma(x) \neq 0 \\ \sigma & \text{otherwise} \end{cases}$$

  - Every function g' : State $\hookrightarrow$ State with $g'(\sigma) = \sigma$ if $\sigma(x) = 0$ is a fixed point for F'

- The functional F_1 has no fixed point if $g_1 \neq g_2$

$$F_1(g) = \begin{cases} g_1 & \text{if } g = g_2 \\ g_2 & \text{otherwise} \end{cases}$$

Fixed point theory for functions

Strict functions

Let $D_\perp =_{def} D \cup \{\perp\}$ with $\perp \notin D$.
A function $f :: D_\perp \Rightarrow D_\perp$ is called *strict* iff $f(\perp) = \perp$.

Complete partial order (CPO)

Let $\mathcal{D} =_{def} D_\perp \Rightarrow D_\perp$ be the set of all strict functions from $D_\perp$ to $D_\perp$ and $f \preceq g$ if $f(x) = g(x)$ or $f(x) = \perp$.
$(\mathcal{D}, \preceq)$ is a *pointed complete partial order* (i.e., every chain in $\mathcal{D}$ has a least upper bound and $\mathcal{D}$ has a least element)

Continuous functions

Let $\mathcal{D}$ be a CPO; a monotonic function $F :: \mathcal{D} \Rightarrow \mathcal{D}$ is *continuous* iff for every chain X in $\mathcal{D}$:
$F(\bigvee X) = \bigvee \{ F(x) \mid x \in X \}$

Fixed point theorem

Fixed point theorem

If $(\mathcal{D}, \preceq)$ is a pointed CPO and
$F :: \mathcal{D} \Rightarrow \mathcal{D}$ is continuous, then
$\bigvee \{ F^i(\bot) \mid i \geq 0 \}$ is a fixed point of F

Remarks

Denotational semantics for imperative programs

- are based on fixed point theory for function domains
- allow for compositional definitions and related proof techniques