Exercise 1 Calculus of Natural Deduction

We consider the *Genzten-Calculus*, also known as calculus of *natural deduction*. The calculus uses *sequents* (german: *Sequenzen*) of the form $\Gamma \vdash A$. They state that the formula $A$ can be syntactically derived from the set of formulas $\Gamma$. If it is possible to derive such a sequent using only the *rules* of the calculus, starting from the *axioms*, we also know that $A$ is a semantic conclusion from $\Gamma$ (as the calculus is *correct*).

The calculus has only one axiom, which states that every formula can be derived from itself: $A \vdash A$, for all formulas $A$. Additionally, there are various rules to derive new sequents from existing ones:

**Conjunction, Disjunction and Implication (Binary Relations)**

- $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B}$ ($\land I$)
- $\frac{\Gamma \vdash A \quad \Gamma \vdash A \lor B}{\Gamma \vdash A}$ ($\lor I$)
- $\frac{\Gamma \vdash B}{\Gamma \vdash A \rightarrow B}$ ($\rightarrow I$)
- $\frac{\Gamma \vdash A \land B}{\Gamma \vdash A}$ ($\land E_l$)
- $\frac{\Gamma \vdash A \land B}{\Gamma \vdash B}$ ($\land E_r$)
- $\frac{\Gamma \vdash A \lor B \quad \Gamma, A \vdash C}{\Gamma \vdash C}$ ($\lor E$, $\lor I$, $\lor E$)

**Truth Values (Constants), Negation (Unary Relation) and Weakening**

- $\frac{\Gamma \vdash \text{False}}{\Gamma \vdash A}$ (False $E$)
- $\frac{\Gamma, A \vdash \text{False}}{\Gamma \vdash \neg A}$ ($\neg I$)
- $\frac{\Gamma \vdash A \quad \Gamma \vdash \text{False}}{\Gamma \vdash A}$ ($\neg E$)
- $\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$ (W)

**Universal and Existential Quantifiers**

- $\frac{\Gamma \vdash \{a_{new}/x\} A}{\Gamma \vdash \forall x.A}$ ($\forall I$)
- $\frac{\Gamma \vdash \forall x.A}{\Gamma \vdash \{t/x\} A}$ ($\forall E$)
- $\frac{\Gamma \vdash \{t/x\} A}{\Gamma \vdash \exists x.A}$ ($\exists I$)
- $\frac{\Gamma \vdash \exists x.A \quad \Gamma, \{a_{new}/x\} A \vdash C}{\Gamma \vdash C}$ ($\exists E$)

The names of the rules are given on the right side in parenthesis. The *I* is an abbreviation of *Introduction*, *E* of *Elimination* and *W* of *Weakening*. The syntax $\{y/x\} A$ denotes that all unbound occurrences of $x$ in $A$ are replaced by $y$. You have to choose a completely new variable for each $a_{new}$, i.e. it must not appear in any term or formula yet. $t$ on the other hand is allowed to be an arbitrary term.

A proof in the calculus is a tree of rule applications, whose leaves are axioms and whose root is the theorem you want to prove. Usually such a proof is done *backwards*, starting with the theorem and trying to reach the axioms.
a) **Prepare!** Prove the following sequent using the Gentzen-Calculus:

\[ \vdash (a \lor (b \land c)) \rightarrow ((a \lor b) \land (a \lor c)) \]

b) **Prepare!** Prove the following sequent using the Gentzen-Calculus:

\[ \vdash \exists x. \forall y. P(x, y) \rightarrow \forall y. \exists x. P(x, y) \]

c) Write an Isabelle/HOL theory for your proofs from a) and b). A skeleton file to start with looks like this:

```isabelle
theory Sheet1 imports Main begin

lemma Exercise_1_a: "(a \lor (b \land c)) \rightarrow ((a \lor b) \land (a \lor c))"
apply (rule ...)
done

lemma Exercise_1_b: "(\exists x. \forall y. P x y) \rightarrow (\forall y. \exists x. P x y)"
...
end
```

The rules of the Gentzen-Calculus correspond to the following Isabelle/HOL rules:

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<th>Gentzen</th>
<th>Isabelle/HOL</th>
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<td>(\bot E)</td>
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**Exercise 2 Hilbert-Calculus**

The Hilbert-Calculus for propositional logic has only one rule called *modus ponens*:

\[
\frac{P \rightarrow Q}{Q} \quad (\text{MP})
\]

Additionally, there are three axioms:

(A1) \( P \rightarrow (Q \rightarrow P) \)

(A2) \( (P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)) \)

(A3) \( (\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P) \)

A proof in the Hilbert-Calculus is a sequence of formulas, where each formula is either an axiom, an assumption or the result of using modus ponens on two formulas appearing earlier in the sequence. The sequent \( \Gamma \vdash P \) states that there is a proof using only the assumptions from \( \Gamma \), which ends in \( P \).

a) **Prepare!** Proof the sequent \( \vdash b \rightarrow (a \rightarrow a) \) using the Hilbert-Calculus.

b) **Prepare!** Proof the sequent \( \vdash a \lor \neg a \) using the Hilbert-Calculus. (*Hint: Use the rules from the lecture to eliminate the \( \lor \) first.*)

c) **Prepare!** Proof the sequent \( \neg \neg a \vdash a \) using the Hilbert-Calculus.

d) Write an Isabelle/HOL theory for these proofs.