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Midterm Exam Logik

Monday, 19.06.2017

Last name:	
First name:	
Student number:	

Please note:

1. When the exam starts, write your last name, first name and your student number on this sheet. Write your student number on every sheet you use.
2. Check that your exam is complete (13 pages)!
3. You have 100 minutes to solve the exam.
4. Write your solutions in a readable way using ballpoint pens or fountain pens **(do not use pencils or red or green color)**. Unreadable solutions will not be considered.
5. You **must not** use your own sheets of paper. Please keep the exam sheets stapled. Single sheets will not be corrected.
6. If the provided space is not sufficient you can use the back of the sheet or the additional sheets at the end of the exam. When you need even more sheets, you can get them from us. When using different sheets, make it clear where we can find your solution!
7. Only language dictionaries and one double-sided handwritten A4 sheet are allowed for the exam. Using smartphones, smartwatches and other electronic devices is not permitted. You smartphones should be turned off. There should be no bag directly on your place. **Your exam will be graded with 0 points if you do not adhere to these rules or try to cheat in any way. Moreover, attempts to cheat will be reported to the examination office.**
8. Before starting to work on a task, carefully read the complete task description!
9. Usually you can continue solving a subtask, even if you could not solve a previous subtask.
10. During the exam we will check that the data on this sheet is correct. Please keep your student id card ready for this.

Task:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Points:									
Maximum:	12	8	8	5	6	5	8	10	8

Total:	
Maximum:	70

Aufgabe 1 Truth tables and normal forms (__/12 Punkte)

a) Give a formula A in disjunctive normal form (DNF), so that the following table is a truth table for A:

p	q	r	A
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

__/3

b) You are given the formula $B \equiv \big(p \wedge (\neg q \wedge r)\big) \vee (\neg p \wedge \neg r)$. Give a formula B' in conjunctive normal form (CNF), so that $B \models\!\!\!\dashv B'$.

Hint: Use the distributivity $\big(A \vee (B \wedge C) \models\!\!\!\dashv (A \vee B) \wedge (A \vee C)\big)$ several times to transform the formula.

__/3

c) Use induction on n to prove:

For all $n \in \mathbb{N}$ with $n > 0$ and all $A, B_1, \ldots, B_n \in F$:
$A \vee (B_1 \wedge \cdots \wedge B_n) \mathrel{\models\!\!\!\dashv} (A \vee B_1) \wedge \cdots \wedge (A \vee B_n)$

In your proof you can use the known logical equivalences from the lecture, in particular the distributivity of $\vee$ and $\wedge$:
$A \vee (B \wedge C) \mathrel{\models\!\!\!\dashv} (A \vee B) \wedge (A \vee C)$

__/6

Aufgabe 2 Tableaux (__ /8 Punkte)

Use the tableaux-method to check whether the following formula is a tautology:

$$A \equiv \Big(r \wedge \big((r \rightarrow \neg p) \vee (r \rightarrow \neg q)\big)\Big) \rightarrow \big(\neg p \wedge \neg q\big)$$

Aufgabe 3 Davis-Putnam (__ /8 Punkte)

Use the Davis-Putnam method to check whether the following formula is satisfiable:

$$A \equiv (p \lor q \lor \neg r \lor t) \land (\neg t \lor \neg q) \land (p \lor \neg q \lor \neg r) \land r \land (\neg p \lor t \lor \neg r) \land (q \lor \neg t)$$

Aufgabe 4 Resolution / Compactness theorem (__ /5 Punkte)

From the lecture we know that: A formula A in CNF is satisfiable if and only if $A \vdash_{Res} \sqcup$. If we see A as a finite set of clauses, then we also have: A finite set of clauses Σ is satisfiable if and only if $\Sigma \vdash_{Res} \sqcup$.

Use the compactness theorem to show that this statement also holds for infinite sets of clauses.

Aufgabe 5 The deductive system $\mathcal{F}_0$ (__ /6 Punkte)

Let $\mathcal{H}$ be the set of derivability-statements of the form:

$\vdash_{\mathcal{F}_0} \neg\neg A \to A$ (Example 2.12)

$\vdash_{\mathcal{F}_0} (A \to B) \to \big((A \to \neg B) \to \neg A\big)$ (Lemma 6)

$\vdash_{\mathcal{F}_0} (B \to A) \to \big((\neg B \to A) \to A\big)$ (Lemma 7)

Check whether $B_1, \ldots, B_9$ (see below) is a proof with shortcuts for $\neg p, \neg q \to p \vdash_{\mathcal{F}_0} q$ in $\mathcal{F}_0$ with assumptions $\mathcal{H}$. Give an explanation for each correct step (e.g. the axiom being used) and mark incorrect steps.

$B_1 \equiv \neg p$

$B_2 \equiv \neg p \to (\neg q \to \neg p)$

$B_3 \equiv \neg q \to \neg p$

$B_4 \equiv \neg q \to p$

$B_5 \equiv (\neg q \to p) \to \big((\neg q \to \neg p) \to \neg\neg q\big)$

$B_6 \equiv \neg\neg q \to q$

$B_7 \equiv (\neg q \to p) \to \big((\neg q \to \neg p) \to q\big)$

$B_8 \equiv (\neg q \to \neg p) \to q$

$B_9 \equiv q$

A a reminder: :
The axioms of $\mathcal{F}_0$:

Ax1: $A \to (B \to A)$

Ax2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Ax3: $(\neg A \to \neg B) \to (B \to A)$

The rules of $\mathcal{F}_0$:

MP: $\dfrac{A, (A \to B)}{B}$

Aufgabe 6 Gentzen sequent calculus (__ /5 Punkte)

Derive the sequent $\vdash_G \big(p \to (q \wedge \neg q)\big) \to \neg p$ in the Gentzen sequent calculus. For every step, note which axiom or rule you used.

As a reminder:

Ax1: $\Gamma, A \vdash_G A, \Delta$

Ax2: $\Gamma, A, \neg A \vdash_G \Delta$

Ax3: $\Gamma \vdash_G A, \neg A, \Delta$

$\mathbf{R}_{\wedge}$**:** $\dfrac{\Gamma, A, B \vdash_G \Delta}{\Gamma, A \wedge B \vdash_G \Delta}$

$\mathbf{R}_{\vee}$**:** $\dfrac{\Gamma \vdash_G A, B, \Delta}{\Gamma \vdash_G A \vee B, \Delta}$

$\mathbf{R}_{\to 1}$**:** $\dfrac{\Gamma, A \vdash_G \Delta, B}{\Gamma \vdash_G A \to B, \Delta}$

$\mathbf{R}_{\to 2}$**:** $\dfrac{\Gamma \vdash_G A, \Delta; \quad \Gamma, B \vdash_G \Delta}{\Gamma, A \to B \vdash_G \Delta}$

$\mathbf{R}_{\neg 1}$**:** $\dfrac{\Gamma, A \vdash_G \Delta}{\Gamma \vdash_G \neg A, \Delta}$

$\mathbf{R}_{\neg 2}$**:** $\dfrac{\Gamma \vdash_G A, \Delta}{\Gamma, \neg A \vdash_G \Delta}$

$\mathbf{R}_{\wedge'}$**:** $\dfrac{\Gamma \vdash_G A, \Delta; \quad \Gamma \vdash_G B, \Delta}{\Gamma \vdash_G A \wedge B, \Delta}$

$\mathbf{R}_{\vee'}$**:** $\dfrac{\Gamma, A \vdash_G \Delta; \quad \Gamma, B \vdash_G \Delta}{\Gamma, A \vee B \vdash_G \Delta}$

Aufgabe 7 Induction / dual formulas (__ /8 Punkte)

In the lecture we defined:

The dual formula $d(A)$ of a formula $A \in F$ is defined as:

$$
\begin{aligned}
d(p) &\equiv p \quad \text{for } p \in V \\
d(\neg A) &\equiv \neg d(A) \\
d(B \vee C) &\equiv d(B) \wedge d(C) \\
d(B \wedge C) &\equiv d(B) \vee d(C).
\end{aligned}
$$

Use structural induction to prove:

Let φ and ψ be variable assignments with $\varphi(p) = 1 - \psi(p)$ for all $p \in V$.
Then for all formulas $A \in F_{\{\neg,\vee,\wedge\}}$: $\mathcal{B}_\varphi\big(d(A)\big) = 1 - \mathcal{B}_\psi(A)$.

This is Lemma 3.20 from the lecture. Therefore you should not use this lemma in your proof.

Aufgabe 8 Semantics of predicate logic (__ /10 Punkte)

Let $S = \left\{\left\{f_{/2}, a_{/0}, b_{/0}, c_{/0}\right\}, \left\{p_{/2}\right\}\right\}$ be a signature and $\mathcal{M} = (D, I)$ mit $D = \left\{d_a, d_b, d_c\right\}$ a S-structure. The function f and the predicate p are interepreted by I as shown in the tables below (the first parameter is on the left and the second parameter at the top):

f	a	b	c
a	b	b	c
b	c	b	c
c	c	c	a

p	a	b	c
a	0	1	0
b	1	1	1
c	0	0	1

For example: $I(f)(d_a, d_b) = d_b$ and $I(f)(d_b, d_a) = d_c$.

For the functions of arity 0 we have $I(a) = d_a$, $I(b) = d_b$, and $I(c) = d_c$.

Moreover let ψ be the variable assignment with $\psi(x) = d_a$ for all variables $x \in V$.

Give the result for each of the evaluations below. For formulas containing quantifiers, give a short explanation of your result.

a) $\mathcal{B}_\psi^\mathcal{M}\left(f(c, f(b, c))\right) =$

b) $\mathcal{B}_\psi^\mathcal{M}\left(p(x, c)\right) =$

c) $\mathcal{B}_\psi^\mathcal{M}\left(\exists x.\forall y.\ p(x, y)\right) =$

d) $\mathcal{B}_\psi^\mathcal{M}\left(\exists x.\forall y.\ p(y, x)\right) =$

e) $\mathcal{B}_\psi^\mathcal{M}\left(\forall y.\exists x.\ p(y, x)\right) =$

f) $\mathcal{B}_\psi^\mathcal{M}\left(\forall x.\ p(a, x) \to \exists y.\ f(x, y) \neq f(y, x)\right) =$

Aufgabe 9 Modelling with predicate logic (__ /8 Punkte)

In this task we consider a domain containing nodes, edges and paths in a directed graph, as well as the element $\perp$. Every element in the domain is of exactly one of these kinds. In particular we do not consider paths to be edges. Intuitively, a path is a (possibly empty) list of edges, in which the end-node of the i-th edge is the start-node of the $(i + 1)$-th edge. To formalize statements about graphs, we define the following predicates and functions:

Predicate	**Meaning**
node(x)	true iff. x is a node
edge(x)	true iff. x is an edge
path(x)	true iff. x is a path

Function	**Meaning**
start(x)	Returns the start-node of edge or path x (or $\perp$, if x is neither edge nor path)
end(x)	Returns the end-node of edge or path x (or $\perp$, if x is neither edge nor path)

Model the following statement in predicate logic using the predicates and functions defined above:

a) For every edge there is an edge going in the opposite direction.

b) At every node a path starts, which ends at the same node.

c) There is a node, which is reachable from every other node via some path.

d) If there is a path from a node A to a node B and an edge from node B to a node C, then there is a path from node A to node C.

Continuation of task ____

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