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Sheet 12: Logik (SS 2017)

Abgabe: Freitag, 07. Juli, 15:30
Abgabekästen neben Raum 34-401.7 (bei AG Softwaretechnik)
Bitte geben Sie zu dritt ab.

Aufgabe 1 Beweise in $\mathcal{F}$

In this task we reuse the abbreviations from sheet 11, task 3.

Give a constructive proof for the following statements. Here "constructive" means that the proof should explain how the wanted derivation can be obtained. In particular you should not use semantic arguments like the completeness of $\mathcal{F}$.

a) Let S be a signature, $\Gamma \subseteq FO(S)$, $A \in FO(S)$, $x \in V$, and $t \in Term(S)$.
If $\Gamma \vdash_{\mathcal{F}} A\{x/t\}$, then $\Gamma \vdash_{\mathcal{F}} \exists x.\ A$

b) Let S be a signature, $\Gamma \subseteq FO(S)$, $A, B \in FO(S)$, and $x \in V$.
If $\Gamma, A \vdash_{\mathcal{F}} B$ and x does not appear as a free variable in $\Gamma \cup \{B\}$, then $\Gamma, \exists x.\ A \vdash_{\mathcal{F}} B$

Aufgabe 2 Eliminierung von Gleichheit

Let $S = (Func, Pred)$ be a signature not containing e. For simplicity we consider the concrete signature with $Func = \{f_{/1}\}$ and $Pred = \{p_{/1}\}$ for this task.

When defining Herbrand theories we considered only formulas in $\mathrm{FO}^{\neq}(S)$, i.e. formulas which do not include equality. In this task we want to see how one can find a formula $A' \in \mathrm{FO}^{\neq}(S')$ for a given formula $A \in \mathrm{FO}(S)$, such that A' is satisfiable if and only if A is satisfiable and where A' uses the extended signature $S' = (Func, Pred')$ with $Pred' = Pred \cup \{e_{/2}\}$. The idea is that the new predicate e should replace equality.

a) Give a closed formula $Eq \in \mathrm{FO}^{\neq}(S')$ that does not use equality. The formula should ensure that the interpretation of e is an equivalence relation (reflexive, symmetric, transitive) in every model for Eq.

b) Give a closed formula $K \in \mathrm{FO}^{\neq}(S')$ that does not use equality. The formula should ensure, that the interpretation of e is a congruence relation, meaning that replacing parameters in predicates or functions with e-equivalent terms should yield equivalent results.

c) Let $A \in \mathrm{FO}(S)$ be a closed formula. Explain how to construct a closed formula $A' \in \mathrm{FO}^{\neq}(S')$ which is satisfiable if and only if A is satisfiable.

Give a short explanation of why there is a model $\mathcal{M}$ for A for every model $\mathcal{M}'$ of A'.

Aufgabe 3 Nicht-Standardmodelle

In the lecture you saw how to prove the existence of a non-standard model for the arithmetic of the natural numbers. In this task we will similarly construct a non-standard model for the arithmetic of the rational numbers.

Let $S = (Func, Pred)$ be the signature with function symbols $Func = \left\{0_{/0}, 1_{/0}, +_{/2}, *_{/2}\right\}$ and predicate symbols $Pred = \left\{<_{/2}\right\}$. Moreover, let $Q = (\mathbb{Q}, I)$ be an S-structure, in which the domain is the set of rational numbers and the symbols 0, 1, +, $*$, and < are interpreted as usual, i.e. $I(0) = 0, I(1) = 1, I(+)(d, e) = d + e$, $I(*)(d, e) = d \cdot e$ and $I(<)(d, e) = 1$ iff $d < e$.

Let $\mathcal{T}_Q$ be the theory of Q, which is the set of all closed formulas over S, for which Q is a model:

$$\mathcal{T}_Q = \left\{A \in \text{FO}_{abg}(S) \;\middle|\; Q \models A\right\} .$$

Now consider the set of formulas

$$\Sigma = \mathcal{T}_Q \cup \left\{0 < x\right\} \cup \left\{A_n \;\middle|\; n \in \mathbb{N}_{>0}\right\}$$

with

$$A_n \equiv \underbrace{(1 + \ldots + 1)}_{n \text{ mal } 1} * x < 1$$

where x is a free variable.

a) Show that Σ is satisfiable.

   Hint: Use the compactness theorem.

b) Show that there is no variable assignment $\psi : V \to \mathbb{Q}$, such that $Q, \psi \models \Sigma$.

c) Two structures $\mathcal{M}, \mathcal{M}'$ over the same signature are called *elementarily equivalent*, if they make the same closed formulas true:
   For all $A \in \text{FO}_{abg}(S)$: $\mathcal{M} \models A$ iff. $\mathcal{M}' \models A$.

   Show that every structure $\mathcal{M}$ which satisfies Σ from subtask a) is elementarily equivalent to Q.

   Hint: Use note 5.12 a).

Comment: Part a) shows that Σ has a model $\mathcal{M}$ and with part c) Q and $\mathcal{M}$ are elementarily equivalent. Part b) essentially sohws that Q is not isomorphic to $\mathcal{M}$. Therefore $\mathcal{M}$ can be called non-standard model.

Aufgabe 4 Herbrand-Modelle

Let $S = \left(\left\{a_{/0}\right\}, \left\{p_{/1}, q_{/2}\right\}\right)$ be a signature and $A \in FO(S)$ with
$A \equiv p(a) \wedge \left(\forall x.\ p(x) \to \left(\exists y.\ q(x, y)\right)\right) \wedge \left(\exists x.\ \neg q(x, a)\right)$.

a) Prove that $A \in FO(S)$ has no Herbrand model.

b) Transform formula A into cleansed prenex normal form (CPF) and then transform it into a Skolem formula. Give your intermediate steps.

c) Let A' be the Skolem formula from part b). Give a signature S' for A', such that $A' \in FO(S')$ (and S' is as small as possible).

d) Describe a Herbrand model for $A' \in FO(S')$.