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Sheet 11: Logik (SS 2017)

Bearbeitung in der Übung am 23./24. Juni

Aufgabe 1 Axiomenschemata von $\mathcal{F}$

Check whether the following formulas are axioms of the deductive System $\mathcal{F}$. In case they are, explain to which of the 6 axiom schemata the formula belongs.

a)
$$\Big(\forall y.\ p(y) \to \big(\forall x.\ q(x,y)\big)\Big) \to \Big(\big(\forall y.\ p(y)\big) \to \big(\forall y.\forall x.\ q(x,y)\big)\Big)$$

b)
$$\Big(\Big(\big(\forall x.\ p(x)\big) \to \big(\forall y.\ p(y)\big)\Big) \to p(z)\Big) \to \Big(\big(\forall x.p(x)\big) \to \Big(\big(\forall y.\ p(y)\big) \to p(z)\Big)\Big)$$

c)
$$\forall y.\big(\forall y.\ p(y) \to p(y)\big) \to \big(p(y) \to p(y)\big)$$

d)
$$\big(\forall x.\ \neg\forall y.\ \neg p(x,y)\big) \to \big(\neg\forall y.\ \neg p(y,y)\big)$$

Aufgabe 2 Beweise im deduktiven System $\mathcal{F}$

Give a proof for the following formulas in the deductive system $\mathcal{F}$. You can use the deduction theorem, generalization theorem, contraposition theoren, and inconsistency rule (see Lemma 5.4).

a)
$$\forall x.\ p(x) \quad \vdash_{\mathcal{F}} \quad \forall y.\ p(y)$$

b)
$$p(c) \quad \vdash_{\mathcal{F}} \quad \neg\forall y.\ \neg p(y)$$

Hints:

1. Use the deduction theorem and generalization theorem before starting the proof (if possible)
2. If the right hand side is a negation of a formula it can help to use the inconsistency rule or the contraposition rule.

Aufgabe 3 Beweise in $\mathcal{F}$ (mit weiteren Abkürzungen)

In this task we extend the deductive system $\mathcal{F}$ with the operators $\vee$ and $\wedge$ and with the $\exists$-quantifier. In this task, we consider these operators only as abbreviations for the existing operators, defined as:

- $\exists x.\ A$ is abbreviation for $\neg\forall x.\ \neg A$
- $A \wedge B$ is abbreviation for $\neg(A \to \neg B)$
- $A \vee B$ is abbreviation for $\neg A \to B$

For example this means that in this task $\exists x.\ A \;\equiv\; \neg\forall x.\ \neg A$ holds (usually only the logical equivalence ($\models\mid$) holds and not the syntactic one ($\equiv$)). However, the abbreviations must be used exactly as defined above. For example $\neg\exists x.\ A \;\not\equiv\; \forall x.\ \neg A$.

Beside the abbreviations for operators we also allow the following abbreviating notation for proofs: If S is a signature, $A_1, \dots, A_n, B \in FO(S)$ are formulas, and $(A_1 \wedge \cdots \wedge A_n) \to B$ or (logically equivalent) $A_1 \to \Big(A_2 \to \big(\cdots \to (A_n \to B)\big)\Big)$ is an axiom of $\mathcal{F}$, then the following rule can be used in proofs[1]:

$$\frac{A_1 \qquad \dots \qquad A_n}{B}$$

Give a proof for the following formulas in the deductive system $\mathcal{F}$. You can use the deduction theorem, generalization theorem, contraposition theoren, and inconsistency rule (see Lemma 5.4).

a)

$$\big(\forall x.\ p(x)\big) \wedge \big(\forall x.\ q(x)\big) \quad \vdash_{\mathcal{F}} \quad \forall x.\ p(x) \wedge q(x)$$

b)

$$(\exists x.\ A) \vee (\exists x.\ B) \quad \vdash_{\mathcal{F}} \quad \exists x.\ A \vee B$$

c)

$$\vdash_{\mathcal{F}} \quad \exists x.\ \Big(p(x) \to \big(\forall x.\ p(x)\big)\Big)$$

[1] In both cases it is easy to derive $A_1, \dots, A_n \vdash_{\mathcal{F}} B$. When using this derivation as assumption, then the definition of “abbreviated proof” from the lecture allows to use the corresponding rule.
For example, $\big(p(x) \wedge q(x)\big) \to p(x)$ is an axiom (schema 1) any yields the following rule in this task:

$$\frac{p(x) \wedge q(x)}{p(x)}$$