Prof. Dr. A. Poetzsch-Heffter
Peter Zeller, M. Sc.

TU Kaiserslautern
Fachbereich Informatik
AG Softwaretechnik

Sheet 5: Logik (SS 2017)

Bearbeitung in der Übung am 11./12. Mai

Aufgabe 1 Generalizing proofs in $\mathcal{F}_0$

Show that proofs in $\mathcal{F}_0$ can be generalized: If there is a proof with variable p, then p can be replaced by an arbitrary formula to get a new valid proof.

To express this statement precisely, we will use the concept of substitutions. If A is a formula, then $A[B/p]$ (say: "A with B for p") is the formula which is obtained by replacing all occurrences of variable p in A by B. Formally we define substitution recursively over the structure of formula:

$$(\neg A)[B/p] \equiv (\neg A[B/p]) \qquad (A_1 \to A_2)[B/p] \equiv (A_1[B/p] \to A_2[B/p]) \qquad p_i[B/p] \equiv \begin{cases} B & \text{falls } p_i \equiv p \\ p_i & \text{falls } p_i \not\equiv p \end{cases}$$

For a set of formulas Σ we define $\Sigma[B/p] = \left\{A[B/p] \,\middle|\, A \in \Sigma\right\}$, i.e. the set in which the substitution has been applied on all formulas from Σ.

With this definition we can now precisely formulate the statement that is to be shown:

$\Sigma \vdash_{\mathcal{F}_0} A$ implies $\Sigma[B/p] \vdash_{\mathcal{F}_0} A[B/p]$ for all $B \in F_0$ and $p \in V$.

Aufgabe 2 The inconsistency rule for $\mathcal{F}_0$

For this task you may use the following assumptions:

For all $A, B, C \in F_0 = F_{\{\neg,\rightarrow\}}$:

$\vdash_{\mathcal{F}_0} A \rightarrow (B \rightarrow A)$	(Axiomenschema 1)
$\vdash_{\mathcal{F}_0} \big(A \rightarrow (B \rightarrow C)\big) \rightarrow \big((A \rightarrow B) \rightarrow (A \rightarrow C)\big)$	(Axiomenschema 2)
$\vdash_{\mathcal{F}_0} (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$	(Axiomenschema 3)
$\vdash_{\mathcal{F}_0} A \rightarrow A$	(Beispiel 2.10: Triviale Implikation)
$\vdash_{\mathcal{F}_0} \neg\neg A \rightarrow A$	(Beispiel 2.12: Doppelnegation entfernen)
$\vdash_{\mathcal{F}_0} (A \rightarrow B) \rightarrow \big((B \rightarrow C) \rightarrow (A \rightarrow C)\big)$	(1: Transitivität der Implikation)
$\vdash_{\mathcal{F}_0} \neg B \rightarrow (B \rightarrow A)$	(2: Folgerung aus Inkonsistenz)
$\vdash_{\mathcal{F}_0} A \rightarrow \neg\neg A$	(3: Doppelnegation einführen)
$\vdash_{\mathcal{F}_0} (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	(4: Kontraposition)
$\vdash_{\mathcal{F}_0} (A \rightarrow B) \rightarrow \big((A \rightarrow \neg B) \rightarrow \neg A\big)$	(6: Negation aus Inkonsistenz / Widerspruchsbeweis)
$\vdash_{\mathcal{F}_0} (B \rightarrow A) \rightarrow \big((\neg B \rightarrow A) \rightarrow A\big)$	(7: Eliminierung von Annahmen / Fallunterscheidung)

a) Show: $\Sigma \vdash_{\mathcal{F}_0} A$ if and only if $\Sigma \cup \{\neg A\}$ is inconsistent in $\mathcal{F}_0$. You can use the deduction theorem and the lemmata given aboven.

   This equivalence is also called **inconsistency rule**.

   Reminder: A set Σ is inconsistent in $\mathcal{F}_0$ if and only if there is a formula B with $\Sigma \vdash_{\mathcal{F}_0} B$ and $\Sigma \vdash_{\mathcal{F}_0} \neg B$.

b) Show the following statements. You can use the lemmata given above, as well as the deduction theorem and the inconsistency rule. However, you must not use semantic arguments like the completeness of $\mathcal{F}_0$.

   1. $\neg(p \rightarrow q) \vdash_{\mathcal{F}_0} q \rightarrow p$
   2. $r \rightarrow \neg(p \rightarrow p) \vdash_{\mathcal{F}_0} \neg r$
   3. $p \rightarrow (q \rightarrow r) \vdash_{\mathcal{F}_0} \neg r \rightarrow (q \rightarrow \neg p)$
   4. $\vdash_{\mathcal{F}_0} p \rightarrow \big(\neg q \rightarrow \neg(p \rightarrow q)\big)$