

# Formalizing Java's Two's-Complement Integral Type in Isabelle/HOL

## ETH Technical Report 458

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November 16, 2004

### **Abstract**

We present a formal model of the Java two's-complement integral arithmetics. The model directly formalizes the arithmetic operations as given in the Java Language Specification (JLS). The algebraic properties of these definitions are derived. Underspecifications and ambiguities in the JLS are pointed out and clarified. The theory is formally analyzed in Isabelle/HOL, that is, machine-checked proofs for the ring properties and divisor/remainder theorems etc. are provided. This work is suited to build the framework for machine-supported reasoning over arithmetic formulae in the context of Java source-code verification.

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# 1 Introduction

Admittedly, modelling numbers in a theorem prover is not really a “sexy subject” at first sight. Numbers are fundamental, well-studied and well-understood, and everyone is used to them since school-mathematics. Basic theories for the naturals, the integers and real numbers are available in all major theorem proving systems (e.g. [GM93a, The02, Pau94]), so why care? However, numbers as specified in a concrete processor or in a concrete programming language semantics are oriented towards an efficient implementation on a machine. They are finite datatypes and typically based on bit-fiddling definitions. Nevertheless, they often possess a surprisingly rich theory (ring properties, for example) that also comprises a number of highly non-standard and tricky laws with non-intuitive and subtle preconditions.

In the context of program verification tools (such as the B tool [Abr96], KIV [BRS<sup>+</sup>00], LOOP [BJ01], and Jive [MPH00], which directly motivated this work), efficient numerical programs, e.g. square roots, trigonometric functions, fast Fourier transformation or efficient cryptographic algorithms represent a particular challenge. Fortunately, theorem proving technology has matured to a degree that the analysis of realistic machine number specifications for widely-used programming languages such as Java or C now is a routine task [Har99].

With respect to the formalization of integers, we distinguish two approaches:

- (1) the *partial approach*: the arithmetic operations  $+ - * / \%$  are only defined on an interval of (mathematical) integers, and left undefined whenever the result of the operation is outside the interval (c.f. [BS02], which is mainly geared towards this approach).
- (2) the *wrap-around approach*: integers are defined on  $[-2^{n-1} .. 2^{n-1} - 1]$ , where in case of overflow the results of the arithmetic operations are mapped back into this interval through modulo calculations. These numbers can be equivalently realized by bitstrings of length  $n$  in the widely-used two’s-complement representation system [Gol02].

While in the formal methods community there is a widespread reluctance to integrate machine number models and therefore a tendency towards either (infinite) mathematical integers or the first approach (“either remain fully formal but focus on a smaller or simpler language [...]; or remain with the real language, but give up trying to achieve full formality.” [ST99]), we strongly argue in favour of the second approach for the following reasons:

- (1) In a wrap-around implementation, certain properties like “Maxint + 1 = Minint” hold. This has the consequence that crucial algebraic properties such as the associativity law “ $a + (b + c) = (a + b) + c$ ”

hold in the wrap-around approach, but not in the partial approach. The wrap-around approach is therefore more suited for automated reasoning.

- (2) Simply using the mathematical operators on a subset of the mathematical integers does not handle surprising definitions of operators appropriately. E.g. in Java the result of an integer division is always rounded towards zero, and thus the corresponding modulo operation can return negative values. This is unusual in mathematics. Therefore, this naïve approach does not only disregard overflows and underflows but also disregards unconventionally defined operators.
- (3) The Java type `int` is defined in terms of wrap-around in the Java Language Specification [GJSB00], so why should a programmer who strictly complies to it in an efficient program be punished by the lack of formal analysis tools?
- (4) Many parts of the JLS have been analyzed formally — so why not the part concerning number representations? There are also definitions and claimed properties that should be checked; and there are also possible inconsistencies or underspecifications as in all other informal specifications.

As technical framework for our analysis we use Isabelle/HOL and the Isar proof documentation package, whose output is directly used throughout this report, including all proofs. Isabelle [Pau94] is a *generic* theorem prover, i.e. new object logics can be introduced by specifying their syntax and inference rules. Isabelle/HOL is an instance of Isabelle with Church's *higher-order logic* (HOL) [GM93b], a classical logic with equality enriched by total polymorphic higher-order functions. In HOL, induction schemes can be expressed inside the logic, as well as (total) functional programs. Isabelle's methodology for safely building up large specifications is the decomposition of problems into *conservative extensions*. A conservative extension introduces new constants (by *constant definitions*) and types (by *type definitions*) only via axioms of a particular, machine-checked form; a proof that conservative extensions preserve consistency can be found in [GM93b]. Among others, the HOL library provides conservative theories for the logical type *bool*, for the numbers such as *int* and for bitstrings *bin*.

## 1.1 Related Work

The formalization of IEEE floating point arithmetics has attracted the interest of researchers for some time [CnM95, AS95], e.g. leading to concrete, industry strength verification technologies used in Intel's IA64 architecture [Har99].

In hardware verification, it is a routine task to verify two’s complement number operations and their implementations on the gate level. Usually, variants of *binary decision diagrams* are used to represent functions over bit words canonically; thus, if a trusted function representation is identical to one generated from a highly complex implementation, the latter is verified. Meanwhile, addition, multiplication and restricted forms of operations involving division and remainder have been developed [HD99]. Unfortunately, it is known that one variant particularly suited for one operation is inherently intractable for another, etc. Moreover, general division and remainder functions have been proven to be intractable by word-level decision diagrams (WLDD) [SBW02]. For all these reasons, the approach is unsuited to investigate the *theory* of two’s complement numbers: for example, the theorem `JavaInt-div-mod` (see Sec. 15), which involves a mixture of all four operations, can only be proven up to a length of 9 bits, even with leading edge technology WLDD packages<sup>1</sup>.

Amazingly, *formalized theories* of the two’s complement number have only been considered recently; i.e. Fox formalized 32-bit words and the ARM processor for HOL [Fox01], and Bondyfalat developed a (quite rudimentary) bit words theory with division in the AOC project [Bon]. In the context of Java and the JLS, Jacobs [Jac03] presented a fragment of the theory of integral types. This work (like ours) applies to Java Card as well since the models of the four smaller integral types (excluding `long`) of Java and Java Card are identical [Sun00, § 2.2.3.1]. However, although our work is in spirit and scope very similar to [Jac03], there are also significant differences:

- We use standard integer intervals as reference model for the arithmetic operations as well as two’s-complement bitstrings for the bitshift and the bitwise AND, OR, XOR operations (which have not been covered by [Jac03] anyway). Where required, we prove lemmas that show the isomorphy between these two representations.
- While [Jac03] just presents the normal behavior of arithmetic expressions, we also cover the exceptional behavior for expressions like “`x / 0`” by adding a second theory layer with so-called strictness principles (see Sec. 18).
- [Jac03] puts a strong emphasis on widening and narrowing operations which are required for the Java types `short` and `byte`. We currently only concentrate on the type `int` and therefore did not model widening and narrowing yet.

The Java Virtual Machine (JVM) [LY96] has been extensively modelled in the Project Bali [Pus98]. However, the arithmetic operations in this JVM

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<sup>1</sup>Thanks to Marc Herbstritt [Her03] to check this for us!

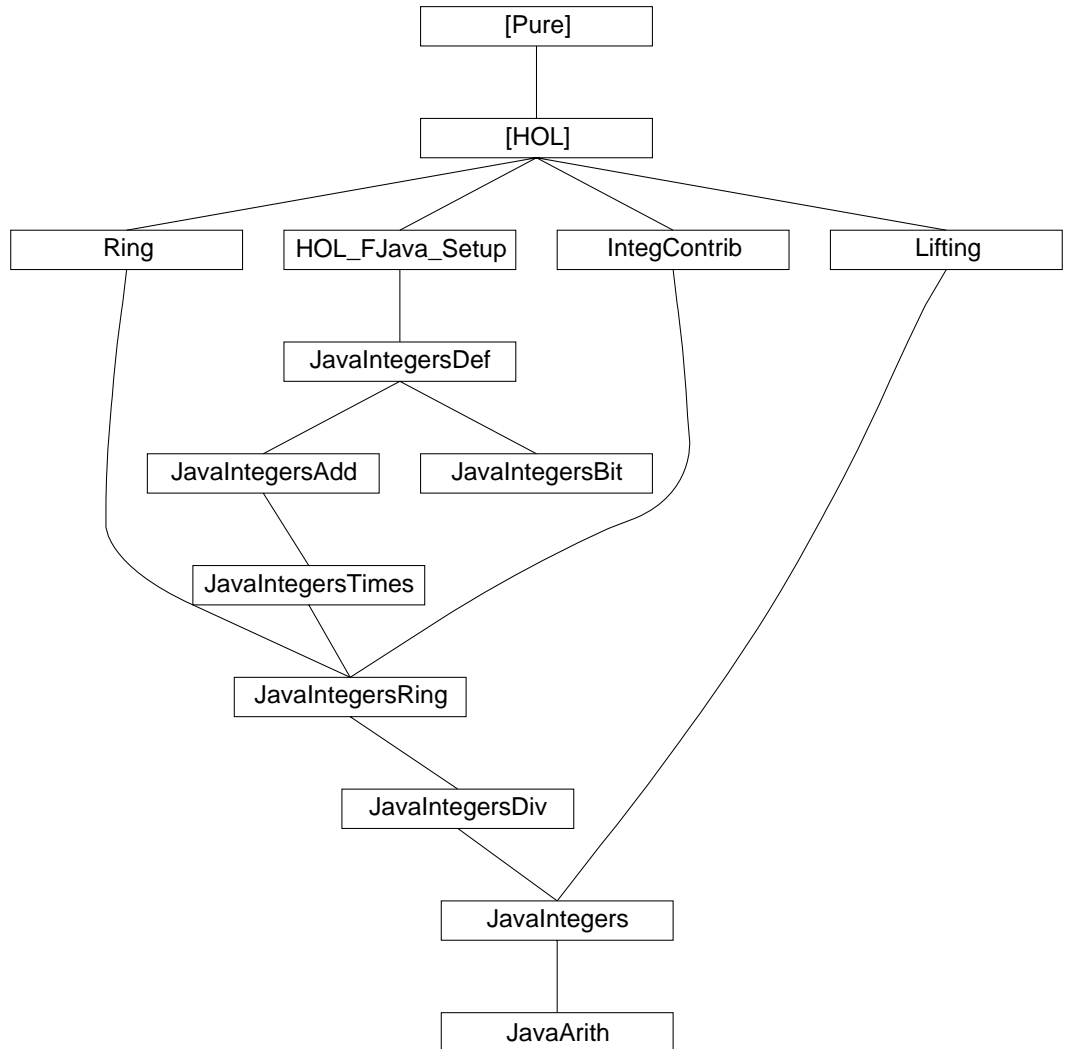
model are based on mathematical integers. Since our work is based on the same system, namely Isabelle2002, our model of a two's-complement integer datatype could replace the mathematical integers in this JVM theory.

## 1.2 Outline of this Report

The specification is divided into two parts: the normal behaviour Java integers (type *JavaInt*) and the exceptional behaviour Java integers (type *java\_int*). In Sec. 2 a graphical overview of the theories is given. After an initial Sec. 3 containing elementary semantic constructions, the first part starts with Sec. 4, which introduces the core conservative definitions of the data type as interval of the “mathematical” integral numbers. Sec. 5 formalizes the addition and Sec. 6 treats the multiplication. Some auxiliary lemmas which belong to Isabelle/HOL's int theory are given in Sec. 11. Ordering properties of *JavaInt* are given in Sec. 12. The ring properties are analysed in Sec. 13. Part one continues with Sec. 14, Sec. 15 and Sec. 16 which present the division and remainder theories and prove some example calculations. Sec. 17 gives a rudimentary theory of bitwise operations based on the library *bits* theory (this section is merely a proof of concept and not further used in this work). Part two is concerned with the exceptional behaviour of Java operators and consists of Sec. 18.



## 2 Overview of the Theories



**theory** *Lifting = Main* :

### 3 Basic Type Construction via Strictification, Smashing and Lifting

The main purpose of this theory is to provide a generic theory of undefinedness. The Isabelle standard mechanism for such a generic data type is the class mechanism. The standard method is to declare a class to which all related operations are associated.

**classes** *bot* < *type*

**consts** *UU* :: '*a*::*bot*

**constdefs** *DEF* :: '*a*::*bot* ⇒ *bool*  
*DEF* *x* ≡ (*x* ≠ *UU*)  
*is-strict* :: ('*a*::*bot* ⇒ '*b*::*bot*) ⇒ *bool*  
*is-strict* *f* ≡ (*f* *UU* = *UU*)  
  
*strictify* :: (('*a*::*bot*) ⇒ ('*b*::*bot*)) ⇒ '*a* ⇒ '*b*  
*strictify* *f* *x* ≡ if *x*=*UU* then *UU* else *f* *x*  
  
*smash* :: [['*b*::*bot*, '*a*::*bot*] ⇒ *bool*, '*a*] ⇒ '*a*  
*smash* *f* *X* ≡ if *f* *UU* *X* then *UU* else *X*

We introduce now the lifting construction (Winskel, pp.132) by a type constructor defined as free data type:

**datatype** '*a* *up* = *lift* '*a* | *down*

**constdefs** *drop* :: '*a* *up* ⇒ '*a*  
*drop* *x* ≡ case *x* of *lift* *v* ⇒ *v* | *down* ⇒ @*x*. *True*

**syntax** @*lift* :: '*a* ⇒ '*a* *up* ( $\lfloor (-) \rfloor$  10)  
@*drop* :: '*a* *up* ⇒ '*a* ( $\lceil (-) \rceil$  10)

**translations**

$\lfloor a \rfloor == \textit{lift } a$   
 $\lceil a \rceil == \textit{drop } a$

The class is then propagated across lifting, function space and cartesian products.

**arithies** *up* :: (*type*) *bot*  
**arithies** *fun* :: (*type*,*bot*) *bot*  
**arithies** \* :: (*bot*,*bot*) *bot*

**defs (overloaded)**  
*UU-up-def* : *UU* ≡ *down*

$UU\text{-fun-def} \quad : \quad UU \equiv (\lambda x. UU)$   
 $UU\text{-pair-def} \quad : \quad UU \equiv (UU, UU)$

**instance**  $up \quad :: (type) \text{ bot } ..$   
**instance**  $fun \quad :: (type, bot) \text{ bot } ..$   
**instance**  $* \quad :: (bot, bot) \text{ bot } ..$

**declare**  $DEF\text{-def} \ [simp] \ UU\text{-up-def} \ [simp] \ UU\text{-fun-def} \ [simp]$

### 3.1 A Generic Theory of Undefinedness, Strictness, Smashing

**lemma**  $not\text{-}DEF\text{-}UU \ [simp]: \neg DEF(UU)$   
**by**  $(simp \ (no\text{-}asm))$

**lemma**  $is\text{-}strict\text{-}strictify:$   
 $is\text{-}strict(strictify \ f)$   
**by**  $(auto \ simp: \ is\text{-}strict\text{-}def \ strictify\text{-}def)$

**lemma**  $strict2a\text{-}UU \ [simp] :$   
 $strictify \ f \ UU = UU$   
**by**  $(simp \ (no\text{-}asm) \ add: \ strictify\text{-}def)$

**lemma**  $strict2b\text{-}UU \ [simp] :$   
 $strictify \ f \ UU \ X = UU$   
**by**  $(simp \ (no\text{-}asm) \ add: \ strictify\text{-}def \ )$

**lemma**  $DEF\text{-}strictify\text{-}DEF\text{-}args2 :$   
 $!!f. \ DEF(strictify \ (\lambda x. \ strictify(f \ x))) \ X \ Y \implies \ DEF \ X \ \& \ DEF \ Y$   
**apply**  $(simp \ add: \ strictify\text{-}def \ )$   
**apply** $(case\text{-}tac \ X=UU)$   
**apply**  $auto$   
**done**

**lemma**  $DEF\text{-}strictify\text{-}DEF\text{-}fun :$   
 $!!f. \ [ \ [!X. \ DEF(f \ X) \ ; \ DEF \ X \ ] \implies \ DEF(strictify \ f \ X)$   
**by**  $(auto \ simp: \ strictify\text{-}def \ )$

**lemma**  $DEF\text{-}strictify\text{-}DEF\text{-}args:$   
 $!!f. \ DEF(strictify \ f \ X) \implies \ DEF \ f \ \& \ DEF \ X$   
**by**  $(auto \ simp: \ strictify\text{-}def \ )$

**lemma**  $is\text{-}strict\text{-}compose :$   
 $!! \ f. \ [ \ [is\text{-}strict \ f; \ is\text{-}strict \ g \ ] \implies \ is\text{-}strict \ (f \ o \ g)$   
**by**  $(auto \ simp: \ is\text{-}strict\text{-}def \ o\text{-}def)$

**lemma**  $smash\text{-}strict \ [simp] :$

*smash f UU = UU*  
**by** (*simp* (*no-asm*) *add: smash-def*)

**lemma** *smashed-sets-nonempty* [*intro!*] :  
 $UU \in \{X. \text{smash } f \ X = X\}$   
**by** (*simp* (*no-asm*))  
— for proofs of non-emptiness of type-definition based on smashed collection types

### 3.2 A Theory of Undefinedness and Strictness in Lifted Types

**lemma** *not-down-exists-lift* :  
 $x \neq \text{down} = (\exists y. x = (\lfloor y \rfloor))$   
**by** (*induct-tac* *x*, *auto*)

**lemma** *not-down-exists-lift2* :  
 $x \neq UU = (\exists y. x = (\lfloor y \rfloor))$   
**by** (*induct-tac* *x*, *auto*)

**lemma** *drop-lift* [*simp*]:  
 $(\lceil \lfloor f \rfloor \rceil) = f$   
**by** (*auto simp: drop-def* )

**lemma** *drop-down* [*simp*]:  
 $(\lceil UU \rceil) = (@ x. \text{True})$   
**by** (*auto simp: drop-def* )

**lemma** *not-DEF-down* [*simp*]:  
 $\neg \text{DEF}(\text{down})$   
**by** (*simp* (*no-asm*))

**lemma** *DEF-lift* [*simp*]:  
 $\text{DEF}(\lfloor x \rfloor)$   
**by** (*simp* (*no-asm*))

**lemma** *DEF-X-up* :  
 $\text{DEF}(X::'a \text{ up}) = (\exists x. X = (\lfloor x \rfloor))$   
**by** (*simp add: not-down-exists-lift* )

**lemma** *not-DEF-X-up* [*simp*]:  
 $(\neg \text{DEF}(X::'a \text{ up})) = (X = UU)$   
**by** (*simp* (*no-asm*))

**lemma** *DEF-fun-lift* [*simp*]:  
 $\text{DEF}(\lambda x. \lfloor (f \ x) \rfloor)$   
**apply** (*simp* (*no-asm*))  
**apply** (*rule notI*)  
**apply** (*drule fun-cong*)  
**apply** *simp*  
**done**

**lemma** *DEF-fun-fun-lift* [*simp*]:  
 $DEF(\lambda x y. \lfloor (f x y) \rfloor)$   
**apply** (*simp* (*no-asm*))  
**apply** (*rule notI*)  
**apply** (*drule fun-cong*) +  
**apply** *simp*  
**done**

**lemma** *drop-lift-idem*:  
 $(\lfloor \lfloor x \rfloor \rfloor) = x$   
**by** (*simp split add: up.split*)

**lemma** *lift-drop-idem*:  
 $\llbracket DEF(x) \rrbracket \implies (\lfloor \lfloor x \rfloor \rfloor) = x$   
**apply** *simp*  
**apply** (*cases x*)  
**apply** (*simp-all add: up.inject*)  
**done**

**lemma** *lift-defined*:  
 $UU \notin X \implies (\lfloor X \rfloor) \neq UU$   
**by** (*auto*)

### 3.3 A Generic Theory of Strictness

**lemma** *strict2a2-UU* [*simp*]:  
 $strictify f down = down$   
**by** (*simp* (*no-asm*) *add: strictify-def*)

**lemma** *strict2c-UU* [*simp*]:  
 $strictify f down X = down$   
**by** (*simp* (*no-asm*) *add: strictify-def*)

**lemma** *strict2d-UU* [*simp*]:  
 $strictify f UU X = UU$   
**by** (*simp* (*no-asm*) *add: strictify-def*)

**lemma** *strict2e-UU* [*simp*]:  
 $strictify (\lambda x. strictify (f x)) (X::'a::bot) UU = UU$   
**by** (*simp add: strictify-def split: split-if*)

**lemma** *strict2f-UU* [*simp*]:  
 $strictify (\lambda x. strictify (f x)) (X::'a::bot) down = down$   
**by** (*simp add: strictify-def split: split-if*)

**lemma** *strict2-DEF* [*simp*]:  
 $\llbracket X \rrbracket. DEF X \implies strictify f X = f X$   
**by** (*auto simp: strictify-def*)

```

lemma strict22-DEF [simp]:
!!X. DEF X  $\implies$  DEF Y  $\implies$ 
strictify ( $\lambda x. \textit{strictify} (f x)) (X::a::bot) Y = f X Y
by(simp add: DEF-X-up strictify-def split: split-if)

declare UU-up-def [simp del] DEF-def [simp del]

end$ 
```

## 4 Formalizing the Normal Behavior Java Integers

**theory** *JavaIntegersDef* = *HOL-FJava-Setup* :

The formalization of Java integers models the primitive Java type `int` as closely as possible. The programming language Java comes with a quite extensive language specification [GJSB00] which tries to be accurate and detailed. Nonetheless, there are several white spots in the Java integer specification which are pointed out in this report. The language Java itself is platform-independent. The bit length of the data type `int` is fixed in a machine-independent way. This simplifies the modelling task. The JLS states about the integral types:

Java Language Specification [GJSB00], §4.2

“The integral types are `byte`, `short`, `int`, and `long`, whose values are 8-bit, 16-bit, 32-bit and 64-bit signed two’s-complement integers, respectively, and `char`, whose values are 16-bit unsigned integers representing Unicode characters. [...] The values of the integral types are integers in the following ranges: [...] For `int`, from  $-2147483648$  to  $2147483647$ , inclusive”

The Java `int` type and its range are formalized in Isabelle/HOL [Pau94] in the following.

First, we fix the bit length. This is the only constant in the whole theory that needs to be adapted if a different bit length (e.g. to represent the Java types `short`, `byte` etc.) is to be represented. Then, we fix the maximum and minimum values that Java integers can have.

**constdefs**

```

BitLength :: nat
BitLength  $\equiv$  32
ld-BitLength :: nat
ld-BitLength  $\equiv$  THE x.  $2^x = \textit{BitLength}$ 
MinInt-int :: int
MinInt-int  $\equiv$   $-(2^{(\textit{BitLength} - 1)})$ 

```

$MaxInt-int :: int$   
 $MaxInt-int \equiv 2^{(BitLength - 1)} - 1$

We introduce the new type of (normal behaviour) Java Integers by the range of the (mathematical) integers as follows:

**typedef**  $JavaInt = \{i. MinInt-int \leq i \ \& \ i \leq MaxInt-int\}$   
**by** (*unfold MinInt-int-def MaxInt-int-def BitLength-def, auto*)

This construct is the Isabelle/HOL shortcut for a type definition which defines the new type  $JavaInt$  isomorphic to the set of integers between  $MinInt-int$  and  $MaxInt-int$ . The isomorphism is established through the automatically provided (total) functions  $Abs-JavaInt :: int \Rightarrow JavaInt$  and  $Rep-JavaInt :: JavaInt \Rightarrow int$  and the two axioms  $Abs-JavaInt-inverse: y \in JavaInt \implies Rep-JavaInt (Abs-JavaInt y) = y$

and  $Rep-JavaInt-inverse: Abs-JavaInt (Rep-JavaInt x) = x$ .

$Abs-JavaInt$  yields an arbitrary value if the argument is outside of the defining interval of  $JavaInt$ .

#### 4.1 General Definitions

We define  $MinInt$  and  $MaxInt$  to be elements of the new type  $JavaInt$ :

**constdefs**  
 $MinInt :: JavaInt$   
 $MinInt \equiv Abs-JavaInt MinInt-int$   
 $MaxInt :: JavaInt$   
 $MaxInt \equiv Abs-JavaInt MaxInt-int$

In Java, calculations are only performed on values of the types `int` and `long`. Values of the three smaller integral types are widened first:

Java Language Specification [GJSB00], §4.2.2

“If an integer operator other than a shift operator has at least one operand of type `long`, then the operation is carried out using 64-bit precision, and the result of the numerical operator is of type `long`. If the other operand is not `long`, it is first widened (§5.1.4) to type `long` by numeric promotion (§5.6). Otherwise, the operation is carried out using 32-bit precision, and the result of the numerical operator is of type `int`. If either operand is not an `int`, it is first widened to type `int` by numeric promotion. The built-in integer operators do not indicate overflow or underflow in any way.”

This report describes the formalization of the Java type `int`, therefore conversions between the different numerical types are not in the focus of this work. The integer types `byte` and `short` can easily be added as all calculations are performed on the type `int` anyways, so the only operations that need to be implemented are the widening to `int`, and the cast operations from `int` to `byte` and `short`, respectively. The Java type `long` can be added equally

easily as our theory uses the bit length as a parameter, so one only need to change the definition of the bit length (see above) to gain a full theory for the Java type `long`, and again only the widening operations need to be added. Therefore, we only concentrate on the Java type `int` in the following. Our model of Java `int` covers all side-effect-free operators. This excludes the operators `++` and `--`, both in pre- and postfix notation. These operators return the value of the variable they are applied to while modifying the value stored in that variable independently from returning the value. We do not treat assignment of any kind either as it represents a side-effect as well. This also disallows combined operators like `a += b` etc. which are a shortcut for `a = a + b`. This is in line with usual specification languages, e.g. JML [LBR99], which also allows only side-effect-free operators in specifications. From a logical point of view, this makes sense as the specification is usually regarded as a set of predicates. In usual logics, predicates are side-effect-free. Thus, expressions with side-effects must be treated differently, either by special Hoare rules or by program transformation.

In our model, all operators are defined in Isabelle/HOL, and their properties as described in the JLS are proven, which ensures the validity of the definitions in our model. In the following, we quote the definitions from the JLS and present the Isabelle/HOL definitions and lemmas.

Our standard approach of defining the arithmetic operators on `Javalnt` is to convert the operands from `Javalnt` to Isabelle `int`, to apply the corresponding Isabelle `int` operation, and to convert the result back to `Javalnt`. The first conversion is performed by the representation function *Rep-JavaInt* (see above). The inverse conversion is performed by the function *Int-to-JavaInt*:

**constdefs**

$$\begin{aligned}
 \textit{Int-to-JavaInt} &:: \textit{int} \Rightarrow \textit{JavaInt} \\
 \textit{Int-to-JavaInt} (x::\textit{int}) &\equiv \\
 &\textit{Abs-JavaInt} ( \\
 &\quad (( x + (-\textit{MinInt-int}) ) \textit{mod} (2 * (-\textit{MinInt-int})) ) \\
 &\quad + \textit{MinInt-int} )
 \end{aligned}$$

This function first adds  $-\textit{MinInt-int}$  to the argument and then performs a modulo calculation by  $2 * -\textit{MinInt-int}$  which maps the value into the interval  $[0 .. 2 * -\textit{MinInt-int} - 1]$  (which is equivalent to only regarding the lowest 32 bits), and finally subtracts the value that was initially added. This definition is identical to the function *Abs-JavaInt* on arguments which are already in *JavaInt*. Larger or smaller values are mapped to *JavaInt* values, extending the domain to `int`.

This standard approach is not followed for operators that are explicitly defined on the bit representation of the arguments. Our approach differs from the approach used by Jacobs [Jac03] who exclusively uses bit representations for the integer representation as well as the operator definitions.



### 4.1.1 General Lemmas

These lemmas are about *MaxInt-int* and *MinInt-int* only.

**lemma** *MaxInt-MinInt-add*:

$$\text{MaxInt-int} + \text{MinInt-int} = -1$$

**by** (*simp add: MaxInt-int-def MinInt-int-def BitLength-def*)

**lemma** *Minustwo-Minus-two* :

$$(-2) * \text{MinInt-int} = - (2 * \text{MinInt-int})$$

**by** *simp*

**lemma** *JavaInt-maxmin-compare*:

$$\text{MaxInt-int} + 1 = - \text{MinInt-int}$$

**by** (*unfold MaxInt-int-def MinInt-int-def BitLength-def, simp*)

**lemma** *JavaInt-maxmin-compare2*:

$$\text{MinInt-int} = - (\text{MaxInt-int} + 1)$$

**by** (*unfold MaxInt-int-def MinInt-int-def BitLength-def, simp*)

**lemma** *JavaInt-maxmin-compare3*:

$$\text{MaxInt-int} = - (\text{MinInt-int} + 1)$$

**by** (*unfold MaxInt-int-def MinInt-int-def BitLength-def, simp*)

**lemma** *null-leq-MaxInt [simp]* :  $0 \leq \text{MaxInt-int}$

**by** (*simp add: MaxInt-int-def BitLength-def*)

**lemma** *MinInt-leq0 [simp]* :  $\text{MinInt-int} \leq 0$

**by** (*simp add: MinInt-int-def BitLength-def*)

**lemma** *MinInt-less0 [simp]*:  $\text{MinInt-int} < 0$

**by** (*unfold MinInt-int-def BitLength-def, simp*)

These lemmas relate *MaxInt-int* and *MinInt-int* to the new *JavaInt* type.

**lemma** *MaxInt-int-in-JavaInt [simp]* :

$$\text{MaxInt-int} : \text{JavaInt}$$

**by** (*simp add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def*)

**lemma** *MinInt-int-in-JavaInt [simp]* :

$$\text{MinInt-int} : \text{JavaInt}$$

**by** (*simp add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def*)

**lemma** *Rep-Abs-remove-general* :

$$\text{Rep-JavaInt} (\text{Abs-JavaInt} (x \text{ mod } - (2 * \text{MinInt-int}) + \text{MinInt-int}))$$

$$= (x \text{ mod } - (2 * \text{MinInt-int}) + \text{MinInt-int})$$

**apply** (*rule Abs-JavaInt-inverse*)

**apply** (*simp add: MinInt-int-def JavaInt-def MaxInt-int-def BitLength-def*)

**apply** (*simp add: zle-add1-eq-le [symmetric] pos-mod-bound*)

**done**

**lemma** *Rep-Abs-remove* :  
 $(\wedge (x::int).$   
 $Rep\text{-}JavaInt (Abs\text{-}JavaInt ((x - MinInt\text{-}int) \bmod - (2 * MinInt\text{-}int) +$   
 $MinInt\text{-}int))$   
 $= ((x - MinInt\text{-}int) \bmod - (2 * MinInt\text{-}int) + MinInt\text{-}int))$   
**apply** (*rule Abs-JavaInt-inverse*)  
**apply** (*simp add: MinInt-int-def JavaInt-def MaxInt-int-def BitLength-def*)  
**apply** (*simp add: zle-add1-eq-le [symmetric] pos-mod-bound*)  
**done**

**lemma** *MinInt-less-minus [simp]* :  $MinInt\text{-}int < - Rep\text{-}JavaInt a$   
**by** (*insert Rep-JavaInt,*  
*simp del: zle-add1-eq-le add: JavaInt-def MinInt-int-def*  
*MaxInt-int-def BitLength-def zle-add1-eq-le [symmetric]*)

**lemma** *MinInt-leq-minus [simp]* :  $MinInt\text{-}int \leq - Rep\text{-}JavaInt a$   
**apply** (*insert MinInt-less-minus*)  
**apply** (*rule zless-or-eq-imp-zle*)  
**apply** *auto*  
**done**

**lemma** *MinInt-leq [simp]* :  $MinInt\text{-}int \leq Rep\text{-}JavaInt b$   
**by** (*insert Rep-JavaInt,*  
*simp add: JavaInt-def MinInt-int-def MaxInt-int-def*  
*BitLength-def*)

**lemma** *MaxInt-geq [simp]* :  $Rep\text{-}JavaInt b \leq MaxInt\text{-}int$   
**by** (*insert Rep-JavaInt,*  
*simp add: JavaInt-def MinInt-int-def MaxInt-int-def*  
*BitLength-def*)

**lemma** *Int-to-JavaInt-charn*:  $Rep\text{-}JavaInt(Int\text{-}to\text{-}JavaInt x) : JavaInt$   
**apply** (*simp add: JavaInt-def*)  
**done**

**lemma** *MinInt-leq-pos [simp]* :  $MinInt\text{-}int \leq Rep\text{-}JavaInt a$   
**by** (*simp add: Rep-JavaInt*)

**lemma** *MinInt-less-pos* :  
 $Rep\text{-}JavaInt a \neq MinInt\text{-}int \implies MinInt\text{-}int < Rep\text{-}JavaInt a$   
**by** (*simp add: int-less-le HOL.neq-commute*)

**lemma** *minus-leq-MaxInt-plus1* [*simp*] :  $\neg \text{Rep-JavaInt } x \leq \text{MaxInt-int} + 1$   
**apply** (*insert MinInt-leq*)  
**apply** (*simp add: MaxInt-int-def BitLength-def JavaInt-def*  
*MinInt-int-def*)  
**done**

**lemma** *minus-leq-MaxInt* : **assumes** *1: Rep-JavaInt x  $\neq$  MinInt-int*  
**shows**  $\neg \text{Rep-JavaInt } x \leq \text{MaxInt-int}$   
**apply** (*simp add: zminus-zle [of Rep-JavaInt x MaxInt-int]*)  
**apply** (*simp add: JavaInt-maxmin-compare3*)  
**apply** (*simp add: add1-zle-eq*)  
**apply** (*cut-tac b = x in MinInt-leq*)  
**apply** (*insert 1 [symmetric]*)  
**apply** (*drule zle-imp-zless-or-eq*)  
**apply** *simp*  
**done**

**lemma** *minus-in-JavaInt2*:  
**assumes** *1: x : JavaInt*  
**assumes** *2: x  $\neq$  MinInt-int*  
**shows**  $x : JavaInt$

**apply** (*simp add: JavaInt-def*)

**apply** (*simp add: zminus-zle [of x MaxInt-int]*)  
**apply** (*simp add: JavaInt-maxmin-compare3*)  
**apply** (*simp add: add1-zle-eq*)

**apply** (*simp add: zle-zminus [of MinInt-int x]*)  
**apply** (*simp add: JavaInt-maxmin-compare[symmetric]*)

**apply** (*insert 1*)  
**apply** (*simp add: JavaInt-def*)  
**apply** (*insert 2*)  
**apply** *auto*  
**done**

**lemma** *minus-in-JavaInt*:  
**assumes** *1: Rep-JavaInt x  $\neq$  MinInt-int*  
**shows**  $\neg \text{Rep-JavaInt } x : JavaInt$   
**by** (*insert 1, simp add: minus-in-JavaInt2 Rep-JavaInt*)

**lemma** *pos-minus-in-JavaInt* [*simp*] :  
**assumes** *1: 0 < Rep-JavaInt b*  
**shows**  $\neg \text{Rep-JavaInt } b : JavaInt$

```

apply (rule minus-in-JavaInt)
apply (insert MinInt-leq0 1)
apply (drule order-le-less-trans [of MinInt-int 0 Rep-JavaInt b],
  simp)
apply simp
done

```

```

lemma NotMinInt1: assumes 1:  $x \neq \text{MinInt}$ 
shows  $\text{abs} (\text{Rep-JavaInt } x) : \text{JavaInt}$ 
apply (insert 1)
apply (simp add: zabs-def)
apply auto
apply (simp-all add: Rep-JavaInt)
apply (simp add: JavaInt-def)
apply (simp add: zminus-zle)
apply (simp add: MinInt-def)
apply (drule Rep-JavaInt-inverse [of x, symmetric, THEN subst])
apply rotate-tac
apply (simp add: Abs-JavaInt-inject MinInt-int-in-JavaInt Rep-JavaInt)
apply (simp add: zminus-zle [of MaxInt-int Rep-JavaInt x])
apply (rule minus-leq-MaxInt, simp)
done

```

```

lemma Not-MinInt-minus-leq-MaxInt:
  assumes 1:  $\text{Rep-JavaInt } x \neq \text{MinInt-int}$ 
shows  $-\text{Rep-JavaInt } x \leq \text{MaxInt-int}$ 
apply (insert 1)
apply (subst zle-add1-eq-le [of  $-\text{Rep-JavaInt } x$  MaxInt-int, symmetric])
apply (simp only: JavaInt-maxmin-compare)
apply (simp only: zminus-zless-zminus)
apply (cut-tac  $b = x$  in MinInt-leq)
apply rotate-tac
apply (drule zle-imp-zless-or-eq, auto)
done

```

```

lemma Abs-Rep-Mod-identity2:
  Abs-JavaInt
  ( $(\text{Rep-JavaInt } a - \text{MinInt-int}) \bmod -(2 * \text{MinInt-int}) + \text{MinInt-int}$ ) = a
apply (rule mod-pos-pos-trivial [symmetric, THEN subst])
apply (subst zle-zdiff-eq, simp)
apply (simp add: zdiff-zless-eq)
apply (simp add: zless-zminus)
apply (simp add: Rep-JavaInt-inverse)
done

```

```

lemma Abs-Rep-Mod-identity-abstract:
  Int-to-JavaInt (Rep-JavaInt x) = x
by (unfold Int-to-JavaInt-def, simp add: Abs-Rep-Mod-identity2)

```

```

lemma Int-to-JavaInt-ident :
  assumes 1:  $x : \text{JavaInt}$ 
  shows  $(\text{Int-to-JavaInt } x) = \text{Abs-JavaInt } (x::\text{int})$ 
proof -
  have Abs-JavaInt
     $((x - \text{MinInt-int}) \bmod (- (2 * \text{MinInt-int})) + \text{MinInt-int})$ 
     $= \text{Abs-JavaInt } ((x - \text{MinInt-int}) + \text{MinInt-int})$ 
  apply (insert 1)
  apply (simp add: JavaInt-def)
  apply auto
  apply (subst mod-pos-pos-trivial)
  apply auto
  apply (simp add: MinInt-int-def MaxInt-int-def)
  done
  also have  $\dots = \text{Abs-JavaInt } x$  by auto
  also from calculation show ?thesis
  by (simp add: Int-to-JavaInt-def JavaInt-def)
qed

```

```

lemma not-MinInt-not-MinInt-int:
shows  $(b \neq \text{MinInt}) = ((\text{Rep-JavaInt } b) \neq \text{MinInt-int})$ 
  by (simp add: Rep-JavaInt-inject[symmetric] MinInt-def Abs-JavaInt-inverse)

```

The following lemmas only talk about elements of *JavaInt*.

```

lemma eq-commute:
   $((a::\text{JavaInt}) = b) = (b = a)$ 
  by (simp add: Rep-JavaInt-inject[symmetric])
  HOL.eq-commute[of Rep-JavaInt a Rep-JavaInt b]

```

```

lemma neq-commute:
   $((a::\text{JavaInt}) \neq b) = (b \neq a)$ 
  by (simp add: Rep-JavaInt-inject[symmetric])
  HOL.neq-commute[of Rep-JavaInt a Rep-JavaInt b]

```

## 4.2 Axclass zero

```

lemma JavaInt-in-zero:
  OFCLASS(JavaInt, zero-class)
apply (rule zero.intro)
..

```

```

instance JavaInt :: zero ..

```

```

defs (overloaded)
  zero-def :  $0 \equiv \text{Abs-JavaInt } 0$ 

```

### 4.2.1 zero Lemmas

```
lemma zero-in-JavaInt [simp] :  
  0 : JavaInt  
  by (unfold JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def, auto)
```

```
lemma Rep-Abs-zero [simp] : Rep-JavaInt (Abs-JavaInt 0) = 0  
  by (simp add: Abs-JavaInt-inverse zero-in-JavaInt)
```

```
lemma Int-to-JavaInt-zero-zero [simp] : Int-to-JavaInt 0 = 0  
  by (simp add: Int-to-JavaInt-ident zero-def)
```

### 4.3 Axclass one

```
lemma JavaInt-in-one:  
  OFCLASS(JavaInt, one-class)  
  apply (rule one.intro)  
  ..
```

```
instance JavaInt :: one ..
```

```
defs (overloaded)  
  one-def : 1  $\equiv$  Abs-JavaInt 1
```

#### 4.3.1 one Lemmas

```
lemma one-in-JavaInt [simp] :  
  1 : JavaInt  
  by (simp add: JavaInt-def one-def MinInt-int-def MaxInt-int-def BitLength-def)
```

### 4.4 Axclass number

Required for bit string representation.

```
instance JavaInt :: number ..
```

```
defs (overloaded)  
  number-of-def : (number-of:: bin  $\Rightarrow$  JavaInt) b  
                  $\equiv$  Int-to-JavaInt ((number-of::bin  $\Rightarrow$  int) b)
```

The inverse operation *bin-of* is defined in Sec. 17 together with the other bitstring operations.

```
end
```

```
theory JavaIntegersAdd = JavaIntegersDef :
```

## 5 Addition and Subtraction

This section formalizes the unary and binary  $+$  and  $-$  operators. The JLS describes the binary  $+$  and  $-$  operators as follows:

Java Language Specification [GJSB00], §15.18.2

“The binary  $+$  operator performs addition when applied to two operands of numeric type, producing the sum of the operands. The binary  $-$  operator performs subtraction, producing the difference of two numeric operands.<sup>(1)</sup> [...] Addition is a commutative operation if the operand expressions have no side effects. Integer addition is associative when the operands are all of the same type<sup>(2)</sup> [...]. If an integer addition overflows, then the result is the low-order bits of the mathematical sum as represented in some sufficiently large two’s-complement format.<sup>(3)</sup> If overflow occurs, then the sign of the result is not the same as the sign of the mathematical sum of the two operand values.<sup>(4)</sup> For both integer and floating-point subtraction, it is always the case that  $a-b$  produces the same result as  $a+(-b)$ .<sup>(5)</sup>”

These two operators are defined in the standard way described above. Their definitions are given below.

### 5.1 Aclass plus

**instance** *JavaInt* :: *plus* ..

This is the definition of the binary  $+$  operator that relates to (1) in the JLS, §15.18.2:

**defs (overloaded)**

*add-def* : (*x*::*JavaInt*) + *y*  $\equiv$   
*Int-to-JavaInt* (*Rep-JavaInt* *x* + *Rep-JavaInt* *y*)

This definition of the addition already captures property (3) in the JLS, §15.18.2.

The unary plus operator on *int* is equivalent to the identity function. This is not very challenging, thus we do not elaborate on this operator.

**constdefs**

*uplus* :: *JavaInt*  $\Rightarrow$  *JavaInt*

*uplus* (*x*::*JavaInt*)  $\equiv$  *x*

#### 5.1.1 plus Lemmas

**lemma** *wraparound-top* [*simp*] : *MaxInt* + 1 = *MinInt*

**proof** –

**have** *MaxInt* + 1 =

```

    Int-to-JavaInt( Rep-JavaInt MaxInt + Rep-JavaInt 1 )
  by (simp add: add-def)
also have ... =
  Int-to-JavaInt(Rep-JavaInt(Abs-JavaInt MaxInt-int) +
    Rep-JavaInt(Abs-JavaInt 1))
  by (simp add: MaxInt-def one-def)
also have ... =
  Int-to-JavaInt(MaxInt-int + 1)
  by (simp add: Abs-JavaInt-inverse)
also have ... = Abs-JavaInt
  ((MaxInt-int + 1 - MinInt-int) mod - (2 * MinInt-int) + MinInt-int)
  by (simp add: Int-to-JavaInt-def)
also have ... = Abs-JavaInt MinInt-int
  by (simp add: MaxInt-int-def MinInt-int-def BitLength-def)
also have ... = MinInt
  by (simp add: MinInt-def [symmetric])
also from calculation show ?thesis .
qed

```

```

lemma JavaInt-add-left-neutral [simp] :
  (0::JavaInt) + x = x
  apply (unfold add-def zero-def)
  apply (subst Abs-JavaInt-inverse)
  apply (simp-all add: Abs-Rep-Mod-identity-abstract)
done

```

This lemma captures the first property described in (2) in the JLS, §15.18.2:

```

lemma JavaInt-add-commute:
  x + y = y + (x::JavaInt)
  by (simp add: add-def zadd-commute)

```

```

lemma Int-to-JavaInt-homom [simp] :
  Int-to-JavaInt (Rep-JavaInt ((Int-to-JavaInt a) + (Int-to-JavaInt b))) =
  Int-to-JavaInt (a + b)
proof -
  have * : Int-to-JavaInt (Rep-JavaInt (Int-to-JavaInt a) + Rep-JavaInt (Int-to-JavaInt
  b)) =
    Int-to-JavaInt (a + b)
  apply (simp add: Int-to-JavaInt-def)
  apply (simp add: Rep-Abs-remove )

  apply (simp add: zadd-assoc [symmetric])
  apply (simp add: zmod-zadd-right-eq [symmetric])
  apply (simp add: zmod-zadd-left-eq [symmetric])
  apply (simp add: zdiff-zadd-eq )
done

```



```

show ?thesis
  apply (simp add: add-def)
  apply (simp add: Rep-JavaInt Int-to-JavaInt-ident
    [of (Rep-JavaInt
      (Int-to-JavaInt (Rep-JavaInt (Int-to-JavaInt a) + Rep-JavaInt (Int-to-JavaInt
        b))))))
  apply (simp add: Rep-JavaInt-inverse *)
  done
qed

```

This lemma captures the second property described in (2) in the JLS, §15.18.2:

```

lemma JavaInt-add-assoc:
   $x + y + z = x + (y + z :: \text{JavaInt})$ 
proof -
  have **: - (2 * MinInt-int) = MaxInt-int + 1 - MinInt-int
    by (simp add: MinInt-int-def MaxInt-int-def BitLength-def)

  have [intro]:  $\bigwedge a b. \text{Int-to-JavaInt } (a + (\text{Rep-JavaInt}(\text{Int-to-JavaInt } b))) = \text{Int-to-JavaInt } (a+b)$ 
  proof -
    fix a b
    have  $\text{Int-to-JavaInt } b = \text{Abs-JavaInt } ((b - \text{MinInt-int}) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int})$ 
    (is - = Abs-JavaInt ?x)
    by (simp add: Int-to-JavaInt-def)
    moreover
    have ?x : JavaInt
    apply (simp add: pos-mod-sign JavaInt-def)
    apply (simp del: zle-add1-eq-le Minustwo-Minus-two add: pos-mod-sign zle-add1-eq-le
      [symmetric] **)
      zless-zdiff-eq [symmetric] pos-mod-bound JavaInt-maxmin-compare)
    done
    ultimately
    have  $\text{Rep-JavaInt}(\text{Int-to-JavaInt } b) = ?x$ 
    by (simp add: Abs-JavaInt-inverse)
    thus ?thesis a b
    apply (simp add: Int-to-JavaInt-def)
    apply (simp only: zmod-zadd-right-eq [symmetric] int-diff-minus-eq zadd-zdiff-eq
      zadd-assoc)
    done
  qed
  moreover
  have  $\bigwedge a b. \text{Int-to-JavaInt } (\text{Rep-JavaInt } (\text{Int-to-JavaInt } a) + b) = \text{Int-to-JavaInt } (a+b)$ 
  proof -
    fix a b
    have  $\text{Int-to-JavaInt } (\text{Rep-JavaInt } (\text{Int-to-JavaInt } a) + b) = \text{Int-to-JavaInt } (b + \text{Rep-JavaInt } (\text{Int-to-JavaInt } a))$ 
    by (simp add:

```

```

zadd-commute)
  also
  have ... = Int-to-JavaInt (b+a) ..
  also
  have ... = Int-to-JavaInt (a+b) by (simp add: zadd-commute)
  also from calculation
  show ?thesis a b .
qed
ultimately
show ?thesis by (unfold add-def, simp add: zadd-assoc)
qed

```

The remaining properties are given in Sec. 12.5 because they require comparison operators which have not been formalized yet.

## 5.2 Aclass minus

**instance** *JavaInt* :: *minus* ..

This is the definition of the binary  $-$  operator that relates to (1) in the JLS, §15.18.2:

```

defs (overloaded)
  diff-def : (x::JavaInt) - y ≡
    Int-to-JavaInt (Rep-JavaInt x - Rep-JavaInt y)

```

In the JLS, the unary minus operator is defined in relation to the binary minus operator described above.

Java Language Specification [GJSB00], §15.15.4

“At run time, the value of the unary minus expression is the arithmetic negation of the promoted value of the operand. For integer values, negation is the same as subtraction from zero.<sup>(1)</sup>  
 [...] negation of the maximum negative int or long results in that same maximum negative number.<sup>(2)</sup>  
 [...] For all integer values x,  $-x$  equals  $(\sim x)+1$ .<sup>(3)</sup>”

The unary minus operator is formalized as

```

defs (overloaded)
  uminus-def : - (x::JavaInt) ≡ Int-to-JavaInt (- Rep-JavaInt x)

```

Note that the unary minus operator has two fixed points: 0 and `MinInt`. This leads to some unexpected results, e.g. `Math.abs(MinInt) = MinInt`, a negative number (see Sec. 12.4 for details). Also, many of the lemmas presented in this report do not hold for `MinInt` and therefore exclude that value in their assumptions.

The bitwise complement operator is defined by unary and binary minus:

“At run time, the value of the unary bitwise complement expression is the bitwise complement of the promoted value of the operand; note that, in all cases,  $\sim x$  equals  $(-x)-1$ .”

This is formalized in Isabelle/HOL as follows:

**constdefs**

```
JavaInt-bitcomplement :: JavaInt  $\Rightarrow$  JavaInt
JavaInt-bitcomplement (x::JavaInt)  $\equiv$  (-x) - (1::JavaInt)
```

For the Java types byte, short or char, the argument is promoted to int first.

We use the notations  $\sim$  and *JavaInt-bitcomplement* interchangeably.

### 5.2.1 minus Lemmas

This is the first property described for the unary minus operator in the JLS:

```
lemma uminus-property: 0 - x = - (x::JavaInt)
by (simp add: uminus-def diff-def zero-def
    Abs-JavaInt-inverse zero-in-JavaInt)
```

**lemma** wraparound-bottom:

```
MinInt - 1 = MaxInt
apply (simp add: diff-def MinInt-def MaxInt-def Int-to-JavaInt-def one-def)
apply (simp add: Abs-JavaInt-inverse)
apply (simp add: MinInt-int-def MaxInt-int-def BitLength-def)
done
```

**lemma** JavaInt-add-minus-inverse:

```
(-(a::JavaInt)) + a = 0
apply (unfold add-def uminus-def)
apply (unfold Int-to-JavaInt-def zero-def)
apply (simp add: Rep-Abs-remove)
apply (simp add: zmod-zadd-left-eq [symmetric])
apply (simp add: MinInt-int-def BitLength-def)
done
```

**lemma** uminus-one-is-minusone [simp]: - (1::JavaInt) = -1

```
apply (simp-all only : JavaIntegersDef.number-of-def JavaIntegersAdd.uminus-def
    JavaIntegersDef.one-def)
apply (subst Abs-JavaInt-inverse)
apply (simp-all)
done
```

This is the second property described for the unary minus operator in the JLS:

**lemma** uminus-MinInt:

```
- MinInt = MinInt
```

```

proof -
  have
    
$$\text{Abs-JavaInt } ((- \text{MinInt-int} - \text{MinInt-int}) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int}) =$$

    
$$\text{Abs-JavaInt } ((- \text{MinInt-int} + (- \text{MinInt-int})) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int})$$

    (is - =  $\text{Abs-JavaInt } ((- ?mi + (- ?mi)) \bmod -(2 * ?mi) + ?mi)$ )
    by auto
  also have
    ... =  $\text{Abs-JavaInt } ((- (?mi + ?mi)) \bmod -(2 * ?mi) + ?mi)$ 
    by (simp only: zminus-zadd-distrib)
  also have
    ... =  $\text{Abs-JavaInt } ((- (2 * ?mi)) \bmod - (2 * ?mi) + ?mi)$ 
    by auto
  also have
    ... =  $\text{Abs-JavaInt } ?mi$ 
    by (simp only: zmod-self, auto)
  also from calculation
  show ?thesis
  apply (simp add: MinInt-def uminus-def Int-to-JavaInt-def Abs-JavaInt-inverse)
  done
qed

```

```

lemma uminus-uminus-ident [simp] :
  - (- (x::JavaInt)) = x
  apply (cases x = MinInt)
  apply (simp add: uminus-MinInt)
  apply (simp add: uminus-def)
  apply (simp add: Rep-JavaInt-inject[symmetric] MinInt-def Abs-JavaInt-inverse)
  apply (simp add: Int-to-JavaInt-ident minus-in-JavaInt)
  apply (simp add: Abs-JavaInt-inverse minus-in-JavaInt)
  apply (simp add: Abs-Rep-Mod-identity-abstract)
  done

```

```

lemma minus-zero [simp] :
  - 0 = (0::JavaInt)
  by (simp add: zero-def uminus-def Rep-JavaInt-inject[symmetric] Abs-JavaInt-inverse)

```

This lemma captures the property described in (5) in the JLS, §15.18.2:

```

lemma diff-uminus:  $a - b = a + (-b::JavaInt)$ 

```

```

proof -
  have Abs-JavaInt
    ((Rep-JavaInt a - Rep-JavaInt b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int) = Abs-JavaInt
    ((Rep-JavaInt a + (-Rep-JavaInt b) - MinInt-int) mod - (2 * MinInt-int)
  +

```

```

      MinInt-int)
    (is - = Abs-JavaInt ((?ra + ?rb - ?m) mod ?mm + ?m))
  by auto
  also have ... =
    Abs-JavaInt ((?ra + (?rb - ?m)) mod ?mm + ?m)
  by (simp only: zadd-zdiff-eq [symmetric])
  also have ... =
    Abs-JavaInt ((?ra + (?rb - ?m) mod ?mm) mod ?mm + ?m)
  by (simp only: zmod-zadd-right-eq [symmetric])
  also have ... =
    Abs-JavaInt ((?ra + ((?rb - ?m) mod ?mm + ?m - ?m)) mod ?mm +
?m)
  by auto
  also have ... =
    Abs-JavaInt ((?ra + ((?rb - ?m) mod ?mm + ?m) - ?m) mod ?mm +
?m)
  by auto
  also from calculation
  show ?thesis
    apply (simp add: uminus-def diff-def add-def Int-to-JavaInt-def)
    apply (simp only: Rep-Abs-remove)
    apply auto
  done
qed

```

```

lemma Int-to-JavaInt-neg-MinInt-int [simp] : Int-to-JavaInt (- MinInt-int) =
MinInt
  apply (simp add: JavaInt-maxmin-compare[symmetric])
  apply (simp add: Int-to-JavaInt-def)
  apply (simp add: MaxInt-int-def MinInt-int-def MinInt-int-def BitLength-def)
  done

```

```

lemma minusone-in-JavaInt [simp] : -1 : JavaInt
  by (simp add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)

```

```

lemma Abs-minusone [simp]: Abs-JavaInt -1 = -1
  by (simp add: number-of-def Int-to-JavaInt-ident)

```

```

lemma Rep-Int-minusone [simp] : Rep-JavaInt (Int-to-JavaInt -1) = -1
  apply (subst Int-to-JavaInt-ident)
  defer
  apply (subst Abs-JavaInt-inverse)
  apply (tactic distinct-subgoals-tac)
  apply (simp add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
  apply simp
  done

```

```

lemma uminus-homom :

```

```

assumes 1:  $a \neq \text{MinInt}$ 
shows  $\text{Rep-JavaInt } (- a) = - \text{Rep-JavaInt } (a)$ 
apply (simp add: uminus-def)
apply (insert 1)
apply (simp add: Rep-JavaInt-inject[symmetric] MinInt-def Abs-JavaInt-inverse)
apply (simp add: Int-to-JavaInt-ident minus-in-JavaInt Abs-JavaInt-inverse)
done

```

```

lemma uminus-shift:
 $(- x = (y::\text{JavaInt})) = (x = - y)$ 
by auto

```

```

lemma uminus-eq-zero-eq-zero [simp]:  $(- xa = (0::\text{JavaInt})) = (xa = 0)$ 
by (simp add: uminus-shift)

```

```

lemma neg-not-MinInt:
 $((- d) \neq \text{MinInt}) = (d \neq \text{MinInt})$ 
by (auto simp: uminus-MinInt uminus-shift)

```

This is the third property described for the unary minus operator in the JLS:

```

lemma uminus-bitcomplement:  $(\text{JavaInt-bitcomplement } x) + 1 = - x$ 

```

```

proof -

```

```

have  $\text{Int-to-JavaInt } (- \text{Rep-JavaInt } x) - 1 + 1 =$ 
 $\text{Int-to-JavaInt } (- \text{Rep-JavaInt } x) + (- 1) + 1$ 
by (simp add: diff-uminus)
also have  $\dots = \text{Int-to-JavaInt } (- \text{Rep-JavaInt } x) + (- 1 + 1)$ 
by (simp only: JavaInt-add-assoc)
also have  $\dots = \text{Int-to-JavaInt } (- \text{Rep-JavaInt } x)$ 
apply (simp only: JavaInt-add-minus-inverse)
apply (simp only: JavaInt-add-commute
 $\text{JavaInt-add-left-neutral}$ )

```

```

done

```

```

also from calculation show ?thesis

```

```

by (simp add: JavaInt-bitcomplement-def uminus-def)

```

```

qed

```

### 5.3 Axclass plus\_ac0

```

instance  $\text{JavaInt} :: \text{plus-ac0}$ 
apply intro-classes
apply (rule JavaInt-add-commute)
apply (rule JavaInt-add-assoc)
apply (rule JavaInt-add-left-neutral)
done

```

**end**

**theory** *JavaIntegersTimes* = *JavaIntegersAdd* :

## 6 Multiplication

### 6.1 Aclass times

**instance** *JavaInt* :: *times* ..

The multiplication operator is described and formalized as follows:

Java Language Specification [GJSB00], §15.17.1

“The binary \* operator performs multiplication, producing the product of its operands. Multiplication is a commutative operation if the operand expressions have no side effects. [...] integer multiplication is associative when the operands are all of the same type”

**defs (overloaded)**

*times-def* :  $x * y \equiv$

*Int-to-JavaInt* (*Rep-JavaInt*  $x * \text{Rep-JavaInt } y$ )

Java Language Specification [GJSB00], §15.17.1

“If an integer multiplication overflows, then the result is the low-order bits of the mathematical product as represented in some sufficiently large two’s-complement format. As a result, if overflow occurs, then the sign of the result may not be the same as the sign of the mathematical product of the two operand values.”

This is again implicitly fulfilled by our standard modelling.

#### 6.1.1 times Lemmas

**lemma** *JavaInt-times-one-ident*:

$1 * (a :: \text{JavaInt}) = a$

**apply** (*unfold one-def times-def*, *simp add: Int-to-JavaInt-def one-def*)

**apply** (*subst Abs-JavaInt-inverse*)

**apply** (*simp add: one-in-JavaInt*)

**apply** (*simp add: Abs-Rep-Mod-identity2*)

**done**

**lemma** *eq-times-eq*:

**assumes**  $1: (x :: \text{JavaInt}) = y$

**shows**  $x * z = y * z$

**apply** (*simp add: times-def*)

**apply** (*insert 1*)

**apply** (*simp only: Rep-JavaInt-inject[symmetric, of x y]*)  
**done**

**lemma** *eq-times-eq2*:  
**assumes**  $1: x = (y::JavaInt)$   
**shows**  $z * x = z * y$   
**apply** (*insert 1*)  
**apply** (*simp add: times-def*)  
**done**

**lemma** *uminus-times-minusone*:  
 $-1 * x = -(x::JavaInt)$   
**by** (*simp add: times-def uminus-def number-of-def*)

The commutativity of the multiplication operator is proven by the lemma

**lemma** *JavaInt-times-commute*:  
 $(x::JavaInt) * y = y * x$   
**by** (*simp add: times-def zmult-commute*)

**lemma** *MinInt-times-minusone* :  
 $MinInt * -1 = MinInt$   
**apply** (*subst JavaInt-times-commute*)  
**apply** (*simp add: uminus-times-minusone uminus-MinInt*)  
**done**

The associativity of the multiplication operator is proven by the lemma

**lemma** *JavaInt-times-assoc*:  
 $x * y * z = x*(y*z::JavaInt)$

**proof** –

{

**fix**  $a b$

**have**

*Abs-JavaInt*

$((a * ((b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int) - MinInt-int)$   
*mod*

$- (2 * MinInt-int) +$

$MinInt-int) =$

*Abs-JavaInt*  $((a * b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int)$

**proof** –

**have** *Abs-JavaInt*

$((a * ((b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int) - MinInt-int)$   
*mod*

$- (2 * MinInt-int) +$

$MinInt-int) = Abs-JavaInt$

$((a * ((b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int) + (-$   
 $MinInt-int)) mod$

$- (2 * MinInt-int) +$



```

    MinInt-int) by auto
also
have ... =
  Abs-JavaInt
    (((a * ((b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int)) mod
- (2 * MinInt-int)
  + (-MinInt-int)) mod
  - (2 * MinInt-int) +
  MinInt-int)
  by (simp only: zmod-zadd-left-eq [symmetric])
also
have ... = Abs-JavaInt
  ((
  (a * ((b - MinInt-int) mod - (2 * MinInt-int)) + a * MinInt-int)
  mod - (2 * MinInt-int) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp only: zadd-zmult-distrib2)
also
have ... = Abs-JavaInt
  ((
  (a * ((b - MinInt-int) mod - (2 * MinInt-int)) mod - (2 * MinInt-int) + a *
MinInt-int)
  mod - (2 * MinInt-int) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp add: zmod-zadd-left-eq [symmetric])
also
have ... = Abs-JavaInt
  ((
  ((a * (b - MinInt-int)) mod - (2 * MinInt-int) + a * MinInt-int)
  mod - (2 * MinInt-int) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp only: zmod-zmult1-eq [symmetric])
also
have ... = Abs-JavaInt
  ((
  ((a * (b - MinInt-int)) + a * MinInt-int)
  mod - (2 * MinInt-int) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp only: zmod-zadd-left-eq [symmetric])
also
have ... = Abs-JavaInt
  ((
  (a * b)
  mod - (2 * MinInt-int) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp add: zdiff-zmult-distrib2)
also
have ... = Abs-JavaInt
  ((
  (a * b) + (-MinInt-int)) mod - (2 * MinInt-int) + MinInt-int)
  by (simp only: zmod-zadd-left-eq [symmetric])
also
from calculation
show ?thesis by auto

```

```

qed
} note Core-times-assoc = this

have **: - (2 * MinInt-int) = MaxInt-int + 1 - MinInt-int
  by (simp add: MinInt-int-def MaxInt-int-def BitLength-def)

have [intro]:  $\bigwedge a b. \text{Int-to-JavaInt } (a * (\text{Rep-JavaInt}(\text{Int-to-JavaInt } b))) =$ 
Int-to-JavaInt (a*b)
proof -
  fix a b
  have Int-to-JavaInt b = Abs-JavaInt ((b - MinInt-int) mod - (2 * MinInt-int)
+ MinInt-int)
    (is - = Abs-JavaInt ?x)
    by (simp add: Int-to-JavaInt-def)
  moreover
  have ?x : JavaInt
    apply (simp add: pos-mod-sign JavaInt-def)
  apply (simp del: zle-add1-eq-le Minustwo-Minus-two add: pos-mod-sign zle-add1-eq-le
[symmetric] **)
    zless-zdiff-eq [symmetric] pos-mod-bound JavaInt-maxmin-compare)
  done
ultimately
have Rep-JavaInt(Int-to-JavaInt b) = ?x
  by (simp add: Abs-JavaInt-inverse)
thus ?thesis a b
  apply (simp add: Int-to-JavaInt-def)
  apply (simp only: Core-times-assoc)
  done
qed

moreover
have  $\bigwedge a b. \text{Int-to-JavaInt } (\text{Rep-JavaInt } (\text{Int-to-JavaInt } a) * b) = \text{Int-to-JavaInt}$ 
(a*b)
proof -
  fix a b
  have Int-to-JavaInt (Rep-JavaInt (Int-to-JavaInt a) * b) =
Int-to-JavaInt (b * Rep-JavaInt (Int-to-JavaInt a)) by (simp add: zmult-commute)
  also
  have ... = Int-to-JavaInt (b*a) ..
  also
  have ... = Int-to-JavaInt (a*b) by (simp add: zmult-commute)
  also from calculation
  show ?thesis a b .
qed
ultimately
show ?thesis by (unfold times-def, simp add: zmult-assoc)
qed

```

The following lemma does not belong to this section thematically, but its

proof is performed very elegantly when multiplication is used, therefore we postponed it to here:

**lemma** *uminus-zero-zero*:

**assumes**  $1: -x = 0$   
**shows**  $x = (0::JavaInt)$   
**apply** (*insert 1*)  
**apply** (*drule eq-times-eq [of - x 0 -1]*)  
**apply** (*simp add: JavaInt-times-commute [of - -1]*)  
**apply** (*simp add: uminus-times-minusone*)  
**done**

**lemma** *JavaInt-add-times-left-distrib*:

$(a + b) * c = a * c + b * (c::JavaInt)$

**proof** –

**have** *Abs-JavaInt* ((  
 ((*Rep-JavaInt* a + *Rep-JavaInt* b – *MinInt-int*) mod – (2 \* *MinInt-int*) +  
*MinInt-int*)  
 \* *Rep-JavaInt* c – *MinInt-int*) mod – (2 \* *MinInt-int*) + *MinInt-int*)  
 =  
*Abs-JavaInt* ((  
 ((*Rep-JavaInt* a + *Rep-JavaInt* b + (– *MinInt-int*)) mod – (2 \* *MinInt-int*)  
 + *MinInt-int*)  
 \* *Rep-JavaInt* c + (– *MinInt-int*)) mod – (2 \* *MinInt-int*) + *MinInt-int*)  
 (is *Abs-JavaInt* (((*?ab* – *?mi*) mod *?m* + *?mi*) \* *?c* – *?mi*) mod *?m* +  
*?mi*)  
 = *Abs-JavaInt* ((*?x* + *?mmi*) mod *?m* + *?mi*)  
**by** *auto*  
**also have** ... =  
*Abs-JavaInt* ((*?x* mod *?m* + *?mmi*) mod *?m* + *?mi*)  
 (is – = *Abs-JavaInt* (((*?y* \* *?c*) mod *?m* + *?mmi*) mod *?m* + *?mi*)  
**by** (*simp only: zmod-zadd-left-eq [symmetric]*)  
**also have** ... =  
*Abs-JavaInt* (((*?y* mod *?m* \* *?c*) mod *?m* + *?mmi*) mod *?m* +  
*?mi*)  
 (is – = *Abs-JavaInt* (((*?abm* mod *?m* + *?mi*) mod *?m* \* *?c*) mod *?m* + *?mmi*  
 ) mod *?m* + *?mi*)  
**by** (*simp only: zmod-zmult1-eq' [symmetric]*)  
**also have** ... =  
*Abs-JavaInt* (((*?abm* + *?mi*) mod *?m* \* *?c*) mod *?m* + *?mmi*) mod *?m*  
 + *?mi*)  
 (is – = *Abs-JavaInt* (((*?ab* + *?mmi* + *?mi*) mod *?m* \* *?c*) mod *?m* + *?mmi*  
 ) mod *?m* + *?mi*)  
**by** (*simp only: zmod-zadd-left-eq [symmetric]*)  
**also have** ... =  
*Abs-JavaInt* (((*?ab* \* *?c*) mod *?m* + *?mmi*) mod *?m* + *?mi*)  
 (is – = *Abs-JavaInt* (((*?a* + *?b*) \* *?c*) mod *?m* + *?mmi*) mod *?m* + *?mi*)  
**by** (*simp only: zmod-zmult1-eq' [symmetric], auto*)  
**also have** ... =

```

  Abs-JavaInt (((?a * ?c + ?b * ?c) mod ?m + ?mmi) mod ?m + ?mi)
  by (simp add: zadd-zmult-distrib)
also have ... =
  Abs-JavaInt (((?a * ?c + ?b * ?c) + ?mmi) mod ?m + ?mi)
  by (simp only: zmod-zadd-left-eq [symmetric])
also have ... =
  Abs-JavaInt ((?mi + ?mmi + ?a * ?c + ?b * ?c + ?mmi) mod ?m + ?mi)
  by auto
also have ... =
  Abs-JavaInt ((?mi + ?a * ?c + ?mmi + ?b * ?c + ?mmi) mod ?m + ?mi)
  by (simp add: zadd-commute)
also have ... =
  Abs-JavaInt ((?mi + ((?a * ?c + ?mmi) + (?b * ?c + ?mmi)) mod ?m) mod
?m + ?mi)
  (is - = Abs-JavaInt ((?mi + (?acm          + ?bcm          ) mod ?m) mod
?m + ?mi))
  by (simp only: zmod-zadd-right-eq [symmetric] zadd-assoc)
also have ... =
  Abs-JavaInt ((?mi + (?acm mod ?m + ?bcm mod ?m) mod ?m) mod ?m +
?mi)
  (is - = Abs-JavaInt ((?mi + ?z          mod ?m) mod ?m + ?mi))
  by (simp only: zmod-zadd1-eq [symmetric])
also have ... = Abs-JavaInt ((?mi + ?z) mod ?m + ?mi)
  by (subst zmod-zadd-right-eq [symmetric], auto)

also from calculation
show ?thesis
  by (simp add: add-def times-def Int-to-JavaInt-def Rep-Abs-remove)
qed

```

end

## 7 The algebraic hierarchy of rings as axiomatic classes

```

theory Ring = Main
files (order.ML):

```

## 8 Constants

Most constants are already declared by HOL.

```

consts
  assoc      :: ['a::times, 'a] => bool          (infixl 50)
  irred      :: 'a::{zero, one, times} => bool
  prime      :: 'a::{zero, one, times} => bool

```

## 9 Axioms

### 9.1 Ring axioms

**axclass** *ring* < *zero, one, plus, minus, times, inverse, power*

*a-assoc*:  $(a + b) + c = a + (b + c)$

*l-zero*:  $0 + a = a$

*l-neg*:  $(-a) + a = 0$

*a-comm*:  $a + b = b + a$

*m-assoc*:  $(a * b) * c = a * (b * c)$

*l-one*:  $1 * a = a$

*l-distr*:  $(a + b) * c = a * c + b * c$

*m-comm*:  $a * b = b * a$

— Definition of derived operations

*minus-def*:  $a - b = a + (-b)$

*inverse-def*: *inverse*  $a =$  (if  $a$  *dvd* 1 then *THE*  $x$ .  $a*x = 1$  else 0)

*divide-def*:  $a / b = a * \text{inverse } b$

*power-def*:  $a \wedge n = \text{nat-rec } 1 (\%u \ b. \ b * a) \ n$

**defs**

*assoc-def*:  $a \text{ assoc } b == a \text{ dvd } b \ \& \ b \text{ dvd } a$

*irred-def*:  $\text{irred } a == a \sim = 0 \ \& \ \sim a \text{ dvd } 1$   
& (ALL  $d$ .  $d \text{ dvd } a \ --> d \text{ dvd } 1 \ | \ a \text{ dvd } d$ )

*prime-def*:  $\text{prime } p == p \sim = 0 \ \& \ \sim p \text{ dvd } 1$   
& (ALL  $a \ b$ .  $p \text{ dvd } (a*b) \ --> p \text{ dvd } a \ | \ p \text{ dvd } b$ )

### 9.2 Integral domains

**axclass**

*domain* < *ring*

*one-not-zero*:  $1 \sim = 0$

*integral*:  $a * b = 0 ==> a = 0 \ | \ b = 0$

### 9.3 Factorial domains

**axclass**

*factorial* < *domain*

*factorial-divisor*: *True*

*factorial-prime*:  $\text{irred } a ==> \text{prime } a$

## 9.4 Euclidean domains

## 9.5 Fields

**axclass**

*field* < *ring*

*field-one-not-zero*:  $1 \sim= 0$

*field-ax*:  $a \sim= 0 \implies a \text{ dvd } 1$

# 10 Basic facts

## 10.1 Normaliser for rings

**use** *order.ML*

**method-setup** *ring* =

```
{* Method.no-args (Method.SIMPLE-METHOD' HEADGOAL (full-simp-tac ring-ss))
*}
{* computes distributive normal form in rings *
```

## 10.2 Rings and the summation operator

**lemma** *natsum-0* [*simp*]:  $\text{setsum } f \ \{..(0::\text{nat})\} = (f \ 0::'a::\text{plus-ac0})$   
**by** *simp*

**lemma** *natsum-Suc* [*simp*]:  
 $\text{setsum } f \ \{..Suc \ n\} = (f \ (Suc \ n) + \text{setsum } f \ \{..n\}::'a::\text{plus-ac0})$   
**by** (*simp add: atMost-Suc*)

**lemma** *natsum-Suc2*:  
 $\text{setsum } f \ \{..Suc \ n\} = (\text{setsum } (\%i. f \ (Suc \ i)) \ \{..n\} + f \ 0::'a::\text{plus-ac0})$   
**proof** (*induct n*)  
  **case** *0* **show** ?*case* **by** *simp*  
**next**  
  **case** *Suc* **thus** ?*case* **by** (*simp add: assoc*)  
**qed**

**lemma** *natsum-cong* [*cong*]:  
 $!!k. \ [j = k; !!i::\text{nat}. i \leq k \implies f \ i = (g \ i::'a::\text{plus-ac0})] \implies$   
 $\text{setsum } f \ \{..j\} = \text{setsum } g \ \{..k\}$   
**by** (*induct j*) *auto*

**lemma** *natsum-zero* [*simp*]:  $\text{setsum } (\%i. 0) \ \{..n::\text{nat}\} = (0::'a::\text{plus-ac0})$   
**by** (*induct n*) *simp-all*

**lemma** *natsum-add* [*simp*]:  
 $!!f::\text{nat} \implies 'a::\text{plus-ac0}.$   
 $\text{setsum } (\%i. f \ i + g \ i) \ \{..n::\text{nat}\} = \text{setsum } f \ \{..n\} + \text{setsum } g \ \{..n\}$

**by** (*induct n*) (*simp-all add: plus-ac0*)

**instance** *ring* < *plus-ac0*

**proof**

**fix** *x y z*

**show** (*x::'a::ring*) + *y* = *y* + *x* **by** (*rule a-comm*)

**show** ((*x::'a::ring*) + *y*) + *z* = *x* + (*y* + *z*) **by** (*rule a-assoc*)

**show** 0 + (*x::'a::ring*) = *x* **by** (*rule l-zero*)

**qed**

**ML** {\*

*local*

*val lhss* =

[*read-cterm* (*sign-of* (*the-context* ()))

(*?t* + *?u::'a::ring*, *TVar* (('z, 0), [])),

*read-cterm* (*sign-of* (*the-context* ()))

(*?t* - *?u::'a::ring*, *TVar* (('z, 0), [])),

*read-cterm* (*sign-of* (*the-context* ()))

(*?t* \* *?u::'a::ring*, *TVar* (('z, 0), [])),

*read-cterm* (*sign-of* (*the-context* ()))

(- *?t::'a::ring*, *TVar* (('z, 0), []))

];

*fun proc sg - t* =

*let val rew* = *Tactic.prove sg* [] []

(*HOLogic.mk-Trueprop*

(*HOLogic.mk-eq* (*t*, *Var* ((*x*, *Term.maxidx-of-term t* + 1), *fastype-of*

*t*))))

(*fn* - => *simp-tac ring-ss 1*)

|> *mk-meta-eq*;

*val* (*t'*, *u*) = *Logic.dest-equals* (*Thm.prop-of rew*);

*in if* *t'* *aconv* *u*

*then None*

*else Some rew*

*end*;

*in*

*val ring-simproc* = *mk-simproc ring lhss proc*;

*end*;

\*}

**ML-setup** {\* *Addsimprocs* [*ring-simproc*] \*}

**lemma** *natsum-ldistr*:

!!*a::'a::ring*. *setsum f* {..*n::nat*} \* *a* = *setsum* (%*i*. *f i* \* *a*) {..*n*}

**by** (*induct n*) *simp-all*

**lemma** *natsum-rdistr*:

!!*a::'a::ring*. *a* \* *setsum f* {..*n::nat*} = *setsum* (%*i*. *a* \* *f i*) {..*n*}

by (induct n) simp-all

### 10.3 Integral Domains

declare one-not-zero [simp]

lemma zero-not-one [simp]:

$0 \sim = (1 :: 'a :: \text{domain})$

by (rule not-sym) simp

lemma integral-iff:

$(a * b = (0 :: 'a :: \text{domain})) = (a = 0 \mid b = 0)$

proof

assume  $a * b = 0$  then show  $a = 0 \mid b = 0$  by (simp add: integral)

next

assume  $a = 0 \mid b = 0$  then show  $a * b = 0$  by auto

qed

lemma m-lcancel:

assumes prem:  $(a :: 'a :: \text{domain}) \sim = 0$  shows conc:  $(a * b = a * c) = (b = c)$

proof

assume eq:  $a * b = a * c$

then have  $a * (b - c) = 0$  by simp

then have  $a = 0 \mid (b - c) = 0$  by (simp only: integral-iff)

with prem have  $b - c = 0$  by auto

then have  $b = b - (b - c)$  by simp

also have  $b - (b - c) = c$  by simp

finally show  $b = c$  .

next

assume  $b = c$  then show  $a * b = a * c$  by simp

qed

lemma m-rcancel:

$(a :: 'a :: \text{domain}) \sim = 0 ==> (b * a = c * a) = (b = c)$

by (simp add: m-lcancel)

end

theory IntegContrib = Main :

## 11 Auxiliary Arithmetic Lemmas

This theory contains auxiliary lemmas on Isabelle/HOL's *int* that should have been proven in the arithmetic library.



The following lemma is used in JavaIntegersRing:

**axioms**

*pos-le-mod-geq*:  $\llbracket (0::int) < b; b \leq a \rrbracket \implies a \text{ mod } b = (a - b) \text{ mod } b$

These lemmas are used in JavaIntegersDiv:

**axioms**

*div-pos-pos-le-pos* :  $\llbracket 0 \leq a; 0 < b \rrbracket \implies (a::int) \text{ div } b \leq a$

*div-neg-pos-ge-pos* :  $\llbracket a < 0; 0 < b \rrbracket \implies a \leq (a::int) \text{ div } b$

*div-notminusone-less* :  $\llbracket a < 0; b < 0 \rrbracket \implies (b::int) \neq -1 \implies a \text{ div } b < (-a)$

*div-pos-pos-gt0* :  $\llbracket b \leq a; 0 < a; 0 < b \rrbracket \implies (0::int) < a \text{ div } b$

**lemma** *divide-both-sides* :

$(x::int) = y \implies x \text{ div } a = y \text{ div } a$

**by** *auto*

**lemma** *div-pos-neg-ge-neg* :

**assumes** 1:  $0 \leq a$

**assumes** 2:  $b < 0$

**shows**  $(-a) \leq (a::int) \text{ div } b$

**apply** (*insert 1*)

**apply** (*cases 0 < a*)

**apply** (*subst zminus-zminus [symmetric, of b]*)

**apply** (*simp only: zdiv-zminus2 [of a - b]*)

**apply** (*rule div-neg-pos-ge-pos*)

**apply** (*simp-all add: 2*)

**apply** (*simp add: zle-def[symmetric]*)

**apply** (*drule zle-anti-sym*)

**apply** *auto*

**done**

**lemma** *div-neg-neg-le-neg* :

**assumes** 1:  $a < 0$

**assumes** 2:  $b < 0$

**shows**  $(a::int) \text{ div } b \leq (-a)$

**apply** (*subst zminus-zminus [symmetric, of b]*)

**apply** (*simp only: zdiv-zminus2 [of a - b]*)

**apply** (*rule div-pos-pos-le-pos*)

**apply** (*simp-all add: zless-or-eq-imp-zle 1 2*)

**done**

**lemma** *div-neg-neg-gt0*:

$\llbracket a \leq b; a < 0; b < 0 \rrbracket \implies (0::int) < a \text{ div } b$

```

apply (simp only: zdiv-zminus-zminus[symmetric, of a b])
apply (rule div-pos-pos-gt0[of -b -a])
apply auto
done

```

```

lemma mult-uminus-shift:
shows  $x * (- y) = (- x) * (y::int)$ 
by auto

```

```

lemma abs-neq0-greater-neq0 :
assumes 1:  $abs(x) \leq abs(y::int)$ 
assumes 2:  $x \neq 0$ 
shows  $y \neq 0 \ \& \ 0 < abs(y)$ 
apply simp
apply (insert zero-le-zabs[of x] )
apply (insert 1)
apply (simp only: zabs-def)
apply (cases  $x < 0$ )
apply (simp only: if-P )
apply (cases  $y < 0$ )
apply (simp only: if-P )
apply (simp only: if-not-P if-False )
apply (cases  $y < 0$ )
apply (simp only: if-P )
apply (simp only: if-not-P if-False )
apply (insert 2)
apply (simp only: HOL.eq-commute[of x 0])
done

```

end

```

theory JavaIntegersRing = JavaIntegersTimes + Ring + IntegContrib :

```

## 12 Ordering Properties

### 12.1 Axclass ord

```

instance JavaInt :: ord ..

```

```

defs (overloaded)

```

```

JavaInt-le-def :  $(x::JavaInt) \leq y \equiv$ 
   $((Rep\ JavaInt\ x) \leq (Rep\ JavaInt\ y))$ 

```

```

JavaInt-less-def :  $(x::JavaInt) < y \equiv$ 
   $x \leq y \ \& \ x \neq y$ 

```

Later, in a world with multiple exceptions, this has to be refined:

**constdefs**

*geq* :: [*JavaInt*, *JavaInt*] ⇒ *bool*

*geq* *x y* ≡ *y* ≤ *x*

*greater* :: [*JavaInt*, *JavaInt*] ⇒ *bool*

*greater* *x y* ≡ *y* < *x*

*equals* :: [*JavaInt*, *JavaInt*] ⇒ *bool*

*equals* *a b* ≡ (*Rep-JavaInt* *a* = *Rep-JavaInt* *b*)

*notequals* :: [*JavaInt*, *JavaInt*] ⇒ *bool*

*notequals* *a b* ≡ ~ (*equals* *a b*)

### 12.1.1 ord Lemmas

**lemma** *MinInt-int-is-least* :

*x* : *JavaInt* ⇒ *MinInt-int* ≤ *x*

**by** (*simp* *add*: *JavaInt-def*)

**lemma** *MinInt-is-least2* :

*MinInt* ≤ *x*

**apply** (*simp* *add*: *MinInt-def*)

**apply** (*simp* *add*: *JavaInt-le-def*)

**apply** (*subst* *Abs-JavaInt-inverse*)

**apply** (*simp* *add*: *MinInt-int-in-JavaInt*)

**apply** (*rule* *MinInt-int-is-least*)

**apply** (*rule* *Rep-JavaInt*)

**done**

**lemma** *MaxInt-int-is-largest* :

*x* : *JavaInt* ⇒ *x* ≤ *MaxInt-int*

**by** (*simp* *add*: *JavaInt-def*)

**lemma** *MaxInt-is-largest2* :

*x* ≤ *MaxInt*

**apply** (*simp* *add*: *MaxInt-def*)

**apply** (*simp* *add*: *JavaInt-le-def*)

**apply** (*subst* *Abs-JavaInt-inverse*)

**apply** (*simp* *add*: *MaxInt-int-in-JavaInt*)

**apply** (*rule* *MaxInt-int-is-largest*)

**apply** (*rule* *Rep-JavaInt*)

**done**

**lemma** *greater-zero-Abs-Rep*: (*0* < *d*) = (*0* < *Rep-JavaInt* *d*)

**by** (*simp* *add*: *int-less-le* *JavaInt-less-def* *JavaInt-le-def* *zero-def*

*Rep-JavaInt-inject*[*symmetric*])

## 12.2 Aclass order

Lemmas required by the axiomatic class *order* are:

```
lemma JavaInt-refl:  $(x::JavaInt) \leq x$   
  by (unfold JavaInt-le-def, auto)
```

```
lemma JavaInt-trans:  $x \leq y \implies y \leq z \implies x \leq (z::JavaInt)$   
  by (unfold JavaInt-le-def, auto)
```

```
lemma JavaInt-antisym:  
   $(x::JavaInt) \leq y \implies y \leq x \implies x = y$   
apply (unfold JavaInt-le-def)  
apply (simp add: Rep-JavaInt-inject [symmetric])  
done
```

```
lemma JavaInt-less-le:  
   $((x::JavaInt) < y) = (x \leq y \ \& \ x \neq y)$   
apply (unfold JavaInt-le-def JavaInt-less-def)  
apply (simp add: Rep-JavaInt-inject [symmetric])  
done
```

```
instance JavaInt :: order  
apply intro-classes  
apply (simp add: JavaInt-refl)  
apply (rule JavaInt-trans)  
apply auto  
apply (rule JavaInt-antisym)  
apply auto  
apply (simp add: JavaInt-less-def)  
apply (simp add: JavaInt-less-def)  
apply (simp add: JavaInt-less-def)  
done
```

## 12.3 Aclass linorder

Lemmas required by the axiomatic class *linorder* are:

```
lemma JavaInt-linear:  
   $(x::JavaInt) \leq y \mid y \leq x$   
  by (unfold JavaInt-le-def, auto)
```

```
instance JavaInt :: linorder  
apply intro-classes  
apply (simp add: JavaInt-linear)  
done
```

### 12.3.1 linorder Lemmas

This is the lowest level of lemmas. It contains characterizing properties of  $<$  and  $\leq$ .

```
lemma MinInt-neq-zero [simp]:  
  shows  $MinInt \neq 0$   
  apply (simp add: MinInt-def zero-def)  
  apply (subst Abs-JavaInt-inject)  
  apply simp-all  
  apply (simp add: MinInt-int-def BitLength-def)  
  done
```

```
lemma zero-neq-MinInt [simp]:  
  shows  $0 \neq MinInt$   
  by (simp add: MinInt-neq-zero[symmetric])
```

```
lemma MaxInt-neq-zero [simp]:  
  shows  $MaxInt \neq 0$   
  apply (simp add: MaxInt-def zero-def)  
  apply (subst Abs-JavaInt-inject)  
  apply simp-all  
  apply (simp add: MaxInt-int-def BitLength-def)  
  done
```

```
lemma zero-neq-MaxInt [simp]:  
  shows  $0 \neq MaxInt$   
  by (simp add: MaxInt-neq-zero[symmetric])
```

```
lemma MaxInt-neq-MinInt [simp]:  
  shows  $MaxInt \neq MinInt$   
  apply (simp add: MaxInt-def MinInt-def)  
  apply (subst Abs-JavaInt-inject)  
  apply simp-all  
  apply (simp add: MinInt-int-def MaxInt-int-def BitLength-def)  
  done
```

```
lemma MinInt-neq-MaxInt [simp]:  
  shows  $MinInt \neq MaxInt$   
  by (simp add: MaxInt-neq-MinInt[symmetric])
```

```
lemma MinInt-le-zero [simp]:  
  shows  $MinInt < 0$   
  apply (simp add: HOL.order.order-less-le)  
  apply (simp add: MinInt-is-least2)  
  done
```

```
lemma zero-le-MaxInt [simp]:  
  shows  $0 < MaxInt$   
  apply (simp add: HOL.order.order-less-le)
```

```

apply (simp add: MaxInt-is-largest2)
done

lemma MaxMin-compare: MinInt-int < MaxInt-int
  by (simp add: MinInt-int-def MaxInt-int-def BitLength-def)

lemma MinInt-le-MaxInt [simp]:
  shows MinInt < MaxInt
  apply (simp add: HOL.order.order-less-le)
  apply (simp add: MaxInt-is-largest2)
  done

lemma MinInt-is-least [simp]:
  shows  $\sim(x < \text{MinInt})$ 
  by (simp add: HOL.linorder-not-less MinInt-is-least2)

lemma MaxInt-is-largest [simp]:
  shows  $\sim(\text{MaxInt} < x)$ 
  by (simp add: HOL.linorder-not-less MaxInt-is-largest2)

lemma Rep-JavaInt-neg-in-JavaInt [simp]:
  assumes 1: MinInt  $\neq$  a
  shows  $- \text{Rep-JavaInt } a : \text{JavaInt}$ 
  apply (rule minus-in-JavaInt)
  apply (insert 1)
  apply (simp add: MinInt-def)
  apply (simp add: Rep-JavaInt-inject[of - a, symmetric])
  apply (simp add: Abs-JavaInt-inverse)
  done

lemma neg-less-zero:
  assumes 1:  $0 \leq a$ 
  shows  $-(a::\text{JavaInt}) \leq 0$ 
proof  $-$ 
  have  $- \text{Rep-JavaInt } a : \text{JavaInt}$ 
  apply (rule Rep-JavaInt-neg-in-JavaInt[of a])
  apply (insert 1)
  apply auto
  apply (insert MinInt-le-zero)
  apply (simp only: HOL.linorder-not-less[symmetric])
  apply auto
  done

then show ?thesis
  apply (simp add: uminus-def)
  apply (simp add: Int-to-JavaInt-ident)
  apply (simp add: JavaInt-le-def zero-def)
  apply (simp add: Abs-JavaInt-inverse)
  apply (insert 1)

```

```

  apply (simp add: zero-def JavaInt-le-def)
done
qed

```

```

lemma neq-MinInt-larger-MinInt:
  (a ≠ MinInt) = (MinInt < a)
  apply (simp add: JavaInt-less-def JavaInt-le-def MinInt-def
    Abs-JavaInt-inverse)
  apply (simp add: JavaIntegersDef.eq-commute [of a - ])
done

```

```

lemma larger-zero-neq-MinInt:
  assumes 1: 0 < a
  shows MinInt ≠ a
  apply (insert MinInt-le-zero 1)
  apply (drule HOL.order-less-trans [of MinInt 0 a], simp)
  apply (simp add: neq-MinInt-larger-MinInt[symmetric])
  apply (simp add: JavaIntegersDef.neq-commute[of a MinInt])
done

```

```

lemma uminus-pos-if-neg:
  assumes 1: a ≠ MinInt
  assumes 2: a < 0
  shows 0 < - a
  apply (simp add: uminus-def)
  apply (insert 1)
  apply (simp add: JavaIntegersDef.neq-commute[of a MinInt])
  apply (simp add: Int-to-JavaInt-ident[of - Rep-JavaInt a]
    Rep-JavaInt-neg-in-JavaInt[of a])
  apply (insert 2)
  apply (simp add: JavaInt-le-def JavaInt-less-def zero-def
    Abs-JavaInt-inverse)
  apply (simp add: Rep-JavaInt-inject[symmetric] Abs-JavaInt-inverse)
done

```

```

lemma uminus-neg-if-pos:
  assumes 1: 0 < (a::JavaInt)
  shows -a < 0
  apply (simp add: uminus-def)
  apply (insert 1)
  apply (frule larger-zero-neq-MinInt)
  apply (simp add: Int-to-JavaInt-ident[of - Rep-JavaInt a]
    Rep-JavaInt-neg-in-JavaInt[of a])
  apply (simp add: JavaInt-le-def JavaInt-less-def zero-def)
  apply (simp add: Rep-JavaInt-inject[symmetric] Abs-JavaInt-inverse)
done

```

```

lemma uminus-neg1:
  assumes 1: - b < 0

```

```

assumes 2:  $b \neq \text{MinInt}$ 
  shows  $0 < (b::\text{JavaInt})$ 
apply (insert 1 2)
apply (simp add: JavaInt-less-def JavaInt-le-def uminus-def zero-def
  Rep-JavaInt-inject[symmetric] MinInt-def Abs-JavaInt-inverse)
apply (simp add: Int-to-JavaInt-ident [of - Rep-JavaInt b]
  minus-in-JavaInt Abs-JavaInt-inverse)
done

```

```

lemma uminus-neg2:
  assumes 1:  $0 < -b$ 
  assumes 2:  $b \neq \text{MinInt}$ 
    shows  $(b::\text{JavaInt}) < 0$ 
  apply (insert 1 2)
apply (simp add: JavaInt-less-def JavaInt-le-def uminus-def zero-def
  Rep-JavaInt-inject[symmetric] MinInt-def)
apply (simp add: Int-to-JavaInt-ident [of - Rep-JavaInt b]
  minus-in-JavaInt Abs-JavaInt-inverse)
done

```

The following lemma is highly important. It relates the  $<$  operation of *int* and *JavaInt*.

```

lemma le-quasi-def:
  shows  $(a < b) = (\text{Rep-JavaInt } a < \text{Rep-JavaInt } b)$ 
apply (simp add: JavaInt-less-def JavaInt-le-def)
apply (simp add: Rep-JavaInt-inject[of a b, symmetric])
apply (simp add: int-less-le[symmetric])
done

```

```

lemma le-neg-bound :  $xa \leq -x \iff \text{Rep-JavaInt } xa \leq -\text{Rep-JavaInt } x$ 
apply (cases x = MinInt)
prefer 2
apply (subst uminus-homom[symmetric])
prefer 2
apply (simp-all add: JavaIntegersRing.JavaInt-le-def uminus-MinInt )
apply (simp add: JavaIntegersRing.JavaInt-le-def[symmetric])
apply (simp add: order-le-less )
apply (rule disjI1)
apply (simp add: JavaIntegersRing.JavaInt-le-def JavaIntegersAdd.uminus-def
  MinInt-def )
apply (simp add: MinInt-int-def BitLength-def )
apply (subst Abs-JavaInt-inverse)
prefer 2
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def )
done

```



```

lemma less-vs-not-le:
  shows  $((z :: JavaInt) \leq w) = (\sim(w < z))$ 
  apply (simp add: HOL.order.order-less-le)
  apply auto
  apply (simp add: HOL.order.order-antisym)
  apply (simp add: JavaInt-le-def)
  done

```

The following lemma is highly important.

```

lemma less-vs-eq-or-le:
  shows  $((z :: JavaInt) \leq w) = ((z = w) \mid (z < w))$ 
  apply (simp add: JavaInt-less-def)
  apply auto
  done

```

```

lemma less-greater-contradict:
  assumes 1:  $x < y$ 
  assumes 2:  $y < (x :: JavaInt)$ 
  shows False
  apply (insert 1 2)
  apply (simp add: JavaInt-less-def JavaInt-le-def Rep-JavaInt-inject[symmetric])
  apply auto
  done

```

```

lemma neg-less-neg-greater-eq:
  assumes 1:  $\sim(x < y)$ 
  assumes 2:  $\sim(y < (x :: JavaInt))$ 
  shows  $x = y$ 
  apply (insert 1 2)
  apply (simp add: JavaInt-less-def JavaInt-le-def Rep-JavaInt-inject[symmetric])
  apply auto
  done

```

```

lemma leq-geq-eq:
  assumes 1:  $b \leq (a :: JavaInt)$ 
  assumes 2:  $a \leq b$ 
  shows  $a = b$ 
  apply (insert 1 2)
  apply (simp add: JavaInt-le-def Rep-JavaInt-inject[symmetric])
  done

```

```

lemma pos-neg-eq:
  assumes 1:  $a \leq 0$ 
  assumes 2:  $-a \leq 0$ 
  assumes 3:  $a \neq MinInt$ 
  shows  $a = (0 :: JavaInt)$ 
  apply (insert 1 2 3)
  apply (simp add: JavaInt-le-def zero-def uminus-def)

```

```

apply (simp add: Rep-JavaInt-inject[symmetric] MinInt-def
  Abs-JavaInt-inverse)
apply (simp add: Int-to-JavaInt-ident minus-in-JavaInt Abs-JavaInt-inverse)
done

```

The following lemma is highly important.

```

lemma weaken-less:
  assumes 1:  $a < b$ 
  shows  $(a::JavaInt) \leq b$ 
apply (insert 1)
apply (simp add: JavaInt-less-def)
done

```

```

lemma strengthen-less:
  assumes 1 :  $a \neq b$ 
  assumes 2 :  $a \leq b$ 
  shows  $(a::JavaInt) < b$ 
apply (insert 1 2)
apply (simp add: less-vs-eq-or-le)
done

```

```

lemma swap-le:
  assumes 1:  $a \neq MinInt$ 
  assumes 2:  $b \neq MinInt$ 
  shows  $(-a < -b) = (b < a)$ 
apply (simp add: uminus-def)
apply (insert 1 2)
apply (simp only: HOL.neq-commute)
apply (simp add: Int-to-JavaInt-ident)
apply (simp add: JavaInt-less-def [of Abs-JavaInt (- Rep-JavaInt a) -] JavaInt-le-def)
apply (simp add: Abs-JavaInt-inverse)
apply (simp add: Abs-JavaInt-inject)
apply (simp add: zminus-equation [of Rep-JavaInt a - Rep-JavaInt b])
apply (simp add: JavaInt-less-def JavaInt-le-def)
apply (simp add: Rep-JavaInt-inject)
done

```

## 12.4 The Absolute Value Function abs

We also need to model the absolute value function on `JavaInt` (in Java, this is provided by the method `public static int java.lang.Math.abs(int a)`):

```

defs (overloaded)
  abs-def :  $abs (x::JavaInt) \equiv Int-to-JavaInt (abs (Rep-JavaInt x))$ 

```

This function has an unexpected property, which is given in the Java API

Specification<sup>2</sup>:

Java API Specification [Jav]

Returns the absolute value of an int value. If the argument is not negative, the argument is returned. If the argument is negative, the negation of the argument is returned.

Note that if the argument is equal to the value of `Integer.MIN_VALUE`, the most negative representable int value, the result is that same value, which is negative.

### 12.4.1 abs Lemmas

To formalize these quite surprising properties, we introduce the following lemmas. They characterize the absolute value function `abs` which was defined above.

```
lemma abs-pos:
  assumes 1:  $0 \leq a$ 
  shows    $abs(a::JavaInt) = a$ 
  apply  (insert 1)
  apply  (simp add: abs-def zabs-def JavaInt-le-def
              zle-def zero-def Abs-Rep-Mod-identity-abstract )
  done
```

```
lemma abs-neg:
  assumes 1:  $a \leq 0$ 
  shows    $abs(a::JavaInt) = -a$ 
  apply  (insert 1)
  apply  (simp add: abs-def zabs-def JavaInt-less-def JavaInt-le-def
              zle-def zero-def uminus-def)
  apply  (auto)
  apply  (simp add: zle-def [symmetric])
  apply  (tactic dtac (int-le-less RS iffD1) 1)
  apply  (tactic dtac (int-le-less RS iffD1) 1)
  apply  (auto)
  done
```

```
lemma abs-MinInt [simp]:
  shows    $abs\ MinInt = MinInt$ 
  apply  (subst abs-neg)
  apply  (auto simp: order-le-less intro: uminus-MinInt MinInt-le-zero)
  done
```

```
lemma abs-MaxInt [simp]:
  shows    $abs\ MaxInt = MaxInt$ 
  apply  (subst abs-pos)
```

---

<sup>2</sup> [http://java.sun.com/j2se/1.4.1/docs/api/java/lang/Math.html#abs\(int\)](http://java.sun.com/j2se/1.4.1/docs/api/java/lang/Math.html#abs(int))

```

apply (auto intro:weaken-less)
done

```

```

lemma abs-zero [simp]:
  shows abs (0::JavaInt) = 0
  apply (subst abs-pos)
  apply (auto)
done

```

```

lemma abs-positive:
  assumes 1:  $x \neq \text{MinInt}$ 
  shows  $0 \leq \text{abs } (x::\text{JavaInt})$ 
  apply (cases 0≤x)
  apply (simp add: abs-pos)
  apply (insert 1)
  apply (subst abs-neg)
  apply (simp add: abs-neg less-vs-not-le)
  apply (simp add: less-vs-not-le[symmetric] less-vs-eq-or-le)
  apply (simp add: linorder-not-le)
  apply (cases - x = 0)
  apply auto
  apply (simp-all add: order-le-less)

  apply (cut-tac x = x in MinInt-is-least2)
  apply (rule uminus-pos-if-neg)
  apply (auto simp: order-le-less)
done

```

## 12.5 Formalizing More plus Lemmas

This section contains more lemmas that arise from the Java Language Specification § 15.18.2. They had to be postponed to here because they require comparison operators which had not been formalized in Sec. 5.1.1 yet.

```

lemma Plus-leq-2MaxInt:
   $\text{Rep-JavaInt } a + \text{Rep-JavaInt } b \leq 2 * \text{MaxInt-int}$ 
  proof -
    have * :  $\text{Rep-JavaInt } a + \text{Rep-JavaInt } b \leq \text{MaxInt-int} + \text{Rep-JavaInt } b$ 
      by (rule zadd-zle-mono1, simp add: MaxInt-geq )
    moreover
    have ** :  $\dots \leq \text{MaxInt-int} + \text{MaxInt-int}$ 
      by (rule zadd-zle-mono2, simp add: MaxInt-geq )
    moreover have *** :  $\dots = 2 * \text{MaxInt-int}$ 
      by auto
    ultimately show ?thesis
      by (subst *** [symmetric], blast intro: order-trans)
  qed

```

```

lemma Plus-geq-2MinInt:
   $2 * \text{MinInt-int} \leq \text{Rep-JavaInt } a + \text{Rep-JavaInt } b$ 
  proof –
    have * :  $\text{MinInt-int} + \text{Rep-JavaInt } b \leq \text{Rep-JavaInt } a + \text{Rep-JavaInt } b$ 
      by (rule zadd-zle-mono1, simp add: MinInt-leq )
    moreover
      have ** :  $\text{MinInt-int} + \text{MinInt-int} \leq \text{MinInt-int} + \text{Rep-JavaInt } b$ 
        by (rule zadd-zle-mono2, simp add: MinInt-leq )
      moreover have *** :  $\text{MinInt-int} + \text{MinInt-int} = 2 * \text{MinInt-int}$ 
        by auto
      ultimately show ?thesis
        by (subst *** [symmetric], blast intro: order-trans )
  qed

```

This lemma captures the property described in (4) in the JLS, §15.18.2:

```

lemma JavaInt-add-overflow-sign :
  assumes 1:  $\text{MaxInt-int} < \text{Rep-JavaInt } a + \text{Rep-JavaInt } b$ 
  shows  $a + b < 0$ 
  proof –

  have upperbound:  $\text{Rep-JavaInt } a + \text{Rep-JavaInt } b - \text{MinInt-int} + (2 * \text{MinInt-int})$ 
    <  $-(2 * \text{MinInt-int})$ 
    apply (simp add: JavaInt-maxmin-compare2)
    apply (cut-tac a = a and b = b in Plus-leq-2MaxInt)
    apply (simp only: zle-add1-eq-le [symmetric])
    apply (subgoal-tac  $2 * \text{MaxInt-int} + 1 < 3 + 3 * \text{MaxInt-int}$ )
    apply (blast intro: zless-trans)
    apply (simp add: MaxInt-int-def BitLength-def)
    done

  also
  have lowerbound:  $0 \leq \text{MinInt-int} + (\text{Rep-JavaInt } a + \text{Rep-JavaInt } b)$ 
    apply (simp only: JavaInt-maxmin-compare2)
    apply (simp only: zadd-commute [of  $-(\text{MaxInt-int} + 1)$  Rep-JavaInt a +
  Rep-JavaInt b])
    apply (simp only: zdiff-def [symmetric])
    apply (simp add: zle-zdiff-eq [of 0 Rep-JavaInt a + Rep-JavaInt b ( $\text{MaxInt-int} + 1$ )])
    apply (simp only: add1-zle-eq 1)
  done

  also
  have A1:  $0 < -(2 * \text{MinInt-int})$ 
    by (simp add: MinInt-int-def BitLength-def)

  moreover
  have A2:  $\text{MinInt-int} + (\text{Rep-JavaInt } a + \text{Rep-JavaInt } b) < -(2 * \text{MinInt-int})$ 
    apply (simp add: upperbound)
    apply (simp only: zadd-commute [of  $3 * \text{MinInt-int}$  -])

```

```

apply (simp only: zless-zdiff-eq [of - - 3 * MinInt-int, symmetric])
apply (simp add: JavaInt-maxmin-compare2)
apply (simp add: add1-zle-eq [symmetric])
apply (cut-tac a = a and b = b in Plus-leq-2MaxInt)
apply (subgoal-tac 2 * MaxInt-int ≤ 2 + 3 * MaxInt-int)
apply (blast intro: order-trans)

apply (simp add: MaxInt-int-def BitLength-def)
done

moreover
have A3: 2 * MinInt-int + (Rep-JavaInt a + Rep-JavaInt b) < 0
  apply (simp only: zadd-commute [of 2 * MinInt-int -])
  apply (simp only: zless-zdiff-eq [of - - 2 * MinInt-int, symmetric])
  apply (simp add: JavaInt-maxmin-compare2)
  apply (cut-tac a = a and b = b in Plus-leq-2MaxInt)
  apply (simp only: zle-add1-eq-le [symmetric])
done

moreover
have mainproof: (Rep-JavaInt a + Rep-JavaInt b - MinInt-int) mod - (2 * MinInt-int) + MinInt-int < 0
  apply (simp add: zless-zdiff-eq [symmetric])
  apply (simp add: pos-le-mod-geq A1 lowerbound)
  apply (simp add: mod-pos-pos-trivial A2 lowerbound)
  apply (simp add: A3)
done

ultimately
show ?thesis
  apply (insert 1)
  apply (simp add: add-def JavaInt-less-def JavaInt-le-def)
  apply (simp add: zero-def Rep-JavaInt-inject [symmetric])
  apply (simp add: Int-to-JavaInt-def)
  apply (simp add: Rep-Abs-remove)
  apply (insert mainproof)
  apply auto
done
qed

```

This is a good example of how inexact several parts of the Java Language Specification are. If indeed only overflow, i.e. regarding two operands whose sum is larger than `MaxInt`, is meant here, then why pose such a complicated question? “the sign of the mathematical sum of the two operand values” will always be positive in this case, so why talk about “the sign of the result is not the same”? It would be much clearer to state “the sign of the result is always negative”. But what if the authors also wanted to describe underflow, i.e. negative overflow, which is sometimes also referred

to as “overflow”? In §4.2.2 the JLS states “The built-in integer operators do not indicate overflow or underflow in any way.” Thus, the term “underflow” is known to the authors and is used in the JLS. Why do they not use it in the context quoted above? This would also explain the complicated phrasing of the above formulation.

To clarify these matters, we add the lemma

```

lemma JavaInt-add-underflow-sign :
  assumes 1: Rep-JavaInt a + Rep-JavaInt b < MinInt-int
  shows  $0 \leq a + b$ 
proof –

have A1:
   $0 \leq \text{Rep-JavaInt } a + \text{Rep-JavaInt } b - \text{MinInt-int} + - (2 * \text{MinInt-int})$ 
  apply simp
  apply (simp only: zadd-commute [of -3 * MinInt-int Rep-JavaInt a + Rep-JavaInt b])
  apply (simp only: zdiff-zle-eq [of - - Rep-JavaInt a + Rep-JavaInt b, symmetric])
  apply simp
  apply (cut-tac a = a and b = b in Plus-geq-2MinInt)
  apply (subgoal-tac - (-3 * MinInt-int) ≤ 2 * MinInt-int)
  apply (blast intro: order-trans)

  apply (simp add: MinInt-int-def BitLength-def)
  done

also
have A2:
   $\text{Rep-JavaInt } a + \text{Rep-JavaInt } b - \text{MinInt-int} + - (2 * \text{MinInt-int}) < - (2 * \text{MinInt-int})$ 
  by (simp add: 1)

also
have mainproof:
   $0 \leq (\text{Rep-JavaInt } a + \text{Rep-JavaInt } b - \text{MinInt-int}) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int}$ 
  apply (simp add: zdiff-zle-eq [symmetric])
  apply (simp only: zmod-zadd-self2 [of Rep-JavaInt a + Rep-JavaInt b - MinInt-int - (2 * MinInt-int), symmetric])
  apply (simp only: mod-pos-pos-trivial A1 A2)
  apply (simp add: Plus-geq-2MinInt)
  done

show ?thesis
  apply (insert 1)
  apply (simp add: add-def zero-def JavaInt-le-def JavaInt-less-def)
  apply (simp add: Int-to-JavaInt-def)
  apply (simp add: Rep-Abs-remove)
  apply (simp only: mainproof)

```

```
done
qed
```

## 13 Ring Properties

Now we can show that normal behaviour `JavaIntegers` have exactly ring properties. There is a slight confusion here, since the Isabelle2003 `Ring` class requires in fact field properties (like “there exists a multiplicative inverse”, but allows to fake them). This will be cleaned up if ported to a more recent Isabelle version.

### 13.1 Axclass power

```
instance JavaInt :: power ..
```

```
defs (overloaded)
```

```
  power-def : (a::JavaInt) ^ n ≡ nat-rec 1 (λu b. b * a) n
```

### 13.2 Axclass inverse

```
instance JavaInt :: inverse ..
```

```
defs (overloaded)
```

```
  inverse-def : inverse (a::JavaInt) ≡ (if a dvd 1 then THE x. a*x = 1 else 0)
```

```
  divide-def : divide (a::JavaInt) b ≡ a * inverse b
```

### 13.3 Axclass ring

```
instance JavaInt :: ring
```

```
apply intro-classes
```

```
apply (rule JavaInt-add-assoc)
```

```
apply (rule JavaInt-add-left-neutral)
```

```
apply (rule JavaInt-add-minus-inverse)
```

```
apply (rule JavaInt-add-commute)
```

```
apply (rule JavaInt-times-assoc)
```

```
apply (rule JavaInt-times-one-ident)
```

```
apply (rule JavaInt-add-times-left-distrib)
```

```
apply (rule JavaInt-times-commute)
```

```
apply (rule diff-uminus)
```

```
apply (simp add: inverse-def one-def)
```

```
apply (simp add: divide-def)
```

```
apply (simp add: power-def one-def)
```

```
done
```



end

**theory** *JavaIntegersDiv* = *JavaIntegersRing* :

## 14 Division Operator

### 14.1 Division Definition

**instance** *JavaInt* :: *Divides.div* ..

In Java, the division operator comes as a surprise if compared to the mathematical definition of division, which is also used in Isabelle/HOL:

Java Language Specification [GJSB00], §15.17.2

“The binary / operator performs division, producing the quotient of its operands. [...] Integer division rounds toward 0. That is, the quotient produced for operands  $n$  and  $d$  that are integers after binary numeric promotion (§5.6.2) is an integer value  $q$  whose magnitude is as large as possible while satisfying  $|d \times q| \leq |n|$ ; moreover,  $q$  is positive when  $|n| \geq |d|$  and  $n$  and  $d$  have the same sign, but  $q$  is negative when  $|n| \geq |d|$  and  $n$  and  $d$  have opposite signs.<sup>(1)</sup> There is one special case that does not satisfy this rule: if the dividend is the negative integer of largest possible magnitude for its type, and the divisor is -1, then integer overflow occurs and the result is equal to the dividend.<sup>(2)</sup> Despite the overflow, no exception is thrown in this case. On the other hand, if the value of the divisor in an integer division is 0, then an `ArithmeticException` is thrown.<sup>(3)</sup>”

This definition points out a major difference between the definition of division in Isabelle/HOL and Java. If the signs of dividend and divisor are different, the results differ by one because Java rounds towards 0 whereas Isabelle/HOL floors the result. Thus, the naïve approach of modelling Java integers by partialization of the corresponding operations of a theorem prover gives the wrong results in these cases.

We model the division by performing case distinctions:

**defs (overloaded)**

```
div-def : (x::JavaInt) div y ≡  
  if ((0 < x & y < 0) | (x < 0 & 0 < y)) & ~ (EX z. Rep-JavaInt x = z *  
  Rep-JavaInt y)  
  then  
    Int-to-JavaInt ( Rep-JavaInt x div Rep-JavaInt y ) + 1  
  else  
    Int-to-JavaInt ( Rep-JavaInt x div Rep-JavaInt y )
```

In a first version of the formalization, we omitted the second condition, namely that the result of Isabelle’s division operator is only to be increased

by 1 if  $y$  is not a factor of  $x$ . We are thankful to Bart Jacobs who pointed us to this flaw (and to the same flaw in the modulo formalization, see below). The properties mentioned in the language report are formalized below. Property (3) is not modelled by the theory presented in this section because this theory does not introduce a bottom element for integers in order to treat exceptional cases. Our model returns 0 in this case due to the definition of the total function `div` in Isabelle/HOL. Exceptions are handled by the next theory layer (see Sec. 18) which adds a bottom element to `JavaInt` and lifts all operations in order to treat exceptions appropriately.

## 14.2 Division Representation Boundedness

**declare** *Abs-JavaInt-inverse* [*simp*]

**lemma** *div-in-JavaInt* :

**assumes** *A*: *Rep-JavaInt*  $a \neq \text{MinInt-int}$  | *Rep-JavaInt*  $b \neq -1$

**shows** *Rep-JavaInt*  $a \text{ div } \text{Rep-JavaInt } b \in \text{JavaInt}$

**proof** –

```
{
  assume 1:  $0 \leq \text{Rep-JavaInt } a$ 
  assume 2:  $\text{Rep-JavaInt } b < 0$ 
  have  $\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b \in \text{JavaInt}$ 
    apply (simp add: JavaInt-def)
    apply (cut-tac  $a = \text{Rep-JavaInt } a$  and  $b = \text{Rep-JavaInt } b$ 
      in div-pos-neg-ge-neg)
    apply (simp-all add: 1 2)
  apply (cut-tac  $a = \text{Rep-JavaInt } a$  and  $b = \text{Rep-JavaInt } b$  in div-nonneg-neg-le0)
  apply (cut-tac 1, simp)
  apply (cut-tac 2, simp)
  apply (insert MinInt-leq-minus[of  $a$ ] null-leq-MaxInt )
  apply (blast intro: order-trans)
  done
} note div-in-JavaInt-pos-neg = this
```

**also**

```
{
  assume 1:  $0 \leq \text{Rep-JavaInt } a$ 
  assume 2:  $0 < \text{Rep-JavaInt } b$ 
  have  $\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b \in \text{JavaInt}$ 
    apply (simp add: JavaInt-def)
    apply (cut-tac  $a = \text{Rep-JavaInt } a$  and  $b = \text{Rep-JavaInt } b$  in
      div-pos-pos-le-pos)
    apply (simp-all add: 1 2)
    apply (insert pos-imp-zdiv-nonneg-iff[of  $\text{Rep-JavaInt } b$   $\text{Rep-JavaInt } a$ ]
      MinInt-leq0 MaxInt-geq[of  $a$ ] 1 2)
    apply (blast intro: order-trans)
  done
} note div-in-JavaInt-nonneg-pos = this
```

```

moreover
{
  assume 1: Rep-JavaInt  $a < 0$ 
  assume 2:  $0 < \text{Rep-JavaInt } b$ 
  have Rep-JavaInt  $a \text{ div } \text{Rep-JavaInt } b \in \text{JavaInt}$ 
  apply (simp add: JavaInt-def)
  apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b in
    div-neg-pos-ge-pos)
  apply (simp-all add: 1 2)
  apply (insert pos-imp-zdiv-neg-iff [of Rep-JavaInt b Rep-JavaInt a]
    MinInt-leq [of a] null-leq-MaxInt 1 2)
  apply (blast intro: order-trans disjI1 [THEN zless-or-eq-imp-zle])
  done
} note div-in-JavaInt-neg-pos = this

moreover
{
  assume 1: Rep-JavaInt  $a < 0$ 
  assume 2: Rep-JavaInt  $b < 0$ 
  assume 3: Rep-JavaInt  $a \neq \text{MinInt-int} \mid \text{Rep-JavaInt } b \neq -1$ 
  have Rep-JavaInt  $a \text{ div } \text{Rep-JavaInt } b \in \text{JavaInt}$ 
  apply (simp add: JavaInt-def)
  apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b in neg-imp-zdiv-nonneg-iff)
  apply (simp add: 2)
  apply (insert 1, rotate-tac)
  apply (drule HOL.order-less-imp-le, simp)
  apply (insert MinInt-leq0)
  apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
    in div-neg-neg-le-neg)
  apply (simp-all add: 1 2)
  apply (cases Rep-JavaInt a  $\neq$  MinInt-int)
  apply (insert Not-MinInt-minus-leq-MaxInt [of a] MinInt-leq0)
  apply (blast intro: order-trans)

  apply (insert 3, simp)
  apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b in div-notminusone-less)
  apply (simp-all add: 1 2)
  apply (simp add: JavaInt-maxmin-compare[symmetric])
  done
} note div-in-JavaInt-neg-neg = this

show ?thesis
  apply (cases  $0 \leq \text{Rep-JavaInt } a$ )
  apply (cases Rep-JavaInt b < 0)
  apply (simp add: div-in-JavaInt-pos-neg)
  apply (cases Rep-JavaInt b = 0)
  apply (simp add: DIVISION-BY-ZERO zero-in-JavaInt)
  apply (rotate-tac -2)

```

```

apply (simp add: zle-def [symmetric])
apply (drule zle-imp-zless-or-eq)
apply (simp add: div-in-JavaInt-nonneg-pos)

apply (simp add: zle-def)
apply (cases 0 < Rep-JavaInt b)
apply (simp add: div-in-JavaInt-neg-pos)
apply (cases Rep-JavaInt b = 0)
apply (simp add: DIVISION-BY-ZERO zero-in-JavaInt)
apply (rotate-tac -2)
apply (simp add: zle-def [symmetric])
apply (drule zle-imp-zless-or-eq)
apply simp
apply (insert A)
apply (simp add: div-in-JavaInt-neg-neg)
done
qed

lemma JavaInt-nondvd-neq-minusone:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b ≠ 0
  shows b ≠ -1
proof –
  show ?thesis
  apply (insert 1, simp add: number-of-def)
  apply (subst Int-to-JavaInt-ident, rule minusone-in-JavaInt)
  apply (erule swap)
  apply (simp only: Abs-JavaInt-inverse minusone-in-JavaInt
    zmod-minus1-right not-not)

  done
qed

lemma JavaInt-nondvd-div-in-JavaInt:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b ≠ 0
  shows Rep-JavaInt a div Rep-JavaInt b ∈ JavaInt
proof –
show ?thesis
  apply (rule div-in-JavaInt)
  apply (insert 1)
  apply (drule JavaInt-nondvd-neq-minusone)
  apply (rule disjI2)
  apply (erule swap)
  apply (simp add: number-of-def)
  apply (subst Int-to-JavaInt-ident, rule minusone-in-JavaInt)
  apply (subst Rep-JavaInt-inject[symmetric])
  apply (simp only: Abs-JavaInt-inverse minusone-in-JavaInt)
  done
qed

```

### 14.3 Division Characterizations

**lemma** *JavaInt-div-nondvd-pos-pos:*

**assumes** 1: *Rep-JavaInt a mod Rep-JavaInt b ≠ 0*

**assumes** 2:  $0 < a$

**assumes** 3:  $0 < b$

**shows**  $((a::JavaInt) \text{ div } b) =$

*Abs-JavaInt (Rep-JavaInt a div Rep-JavaInt b)*

**proof** –

**show** *?thesis*

**apply** (*insert 1,insert 2,insert 3*)

**apply** (*simp add:div-def split del: split-if*)

**apply** (*subst if-not-P*)

**apply** (*simp-all add: Int-to-JavaInt-ident*)

**apply** (*rule disjI1*)

**apply** (*rule conjI*)

**apply** (*rule order-less-not-sym*)

**defer** 1

**apply** (*rule order-less-not-sym*)

**apply** (*simp-all add: Int-to-JavaInt-ident JavaInt-nondvd-div-in-JavaInt*)

**done**

**qed**

**lemma** *JavaInt-div-nondvd-neg-neg:*

**assumes** 1: *Rep-JavaInt a mod Rep-JavaInt b ≠ 0*

**assumes** 2:  $a < 0$

**assumes** 3:  $b < 0$

**shows**  $((a::JavaInt) \text{ div } b) = \text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ div Rep-JavaInt } b)$

**proof** –

**show** *?thesis*

**apply** (*insert 1,insert 2,insert 3*)

**apply** (*simp add:div-def split del: split-if*)

**apply** (*subst if-not-P*)

**apply** (*simp-all add: Int-to-JavaInt-ident*)

**apply** (*rule disjI1*)

**apply** (*rule conjI*)

**apply** (*rule order-less-not-sym*)

**defer** 1

**apply** (*rule order-less-not-sym*)

**apply** (*simp-all add: Int-to-JavaInt-ident JavaInt-nondvd-div-in-JavaInt*)

**done**

**qed**

**lemma** *JavaInt-div-nondvd-pos-neg:*

**assumes** 1: *Rep-JavaInt a mod Rep-JavaInt b ≠ 0*

**assumes** 2:  $0 < a$

**assumes** 3:  $b < 0$

**shows**  $((a::JavaInt) \text{ div } b) = \text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ div Rep-JavaInt } b) + 1$

```

proof –
show ?thesis
  apply (insert 1,insert 2,insert 3)
  apply (simp add:div-def split del: split-if)
  apply (subst if-P)
  apply (simp ! add: zmod-eq-0-iff zero-def)
  apply auto
  apply (simp-all add: Int-to-JavaInt-ident JavaInt-nondvd-div-in-JavaInt)
  done
qed

```

```

lemma JavaInt-div-nondvd-neg-pos:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b ≠ 0
  assumes 2: a < 0
  assumes 3: 0 < b
  shows ((a::JavaInt) div b) = Abs-JavaInt (Rep-JavaInt a div Rep-JavaInt
b) + 1
proof –
show ?thesis
  apply (insert 1,insert 2,insert 3)
  apply (simp add:div-def split del: split-if)
  apply (subst if-P)
  apply (simp-all add: Int-to-JavaInt-ident)

  apply (simp ! add: zmod-eq-0-iff zero-def)
  apply auto
  apply (simp-all add: Int-to-JavaInt-ident JavaInt-nondvd-div-in-JavaInt)
  done
qed

```

## 15 Remainder Operator

### 15.1 Remainder Definition

The remainder operator is closely related to the division operator. Thus, it does not conform to standard mathematical definitions either.

“The binary `%` operator is said to yield the remainder of its operands from an implied division [...]. The remainder operation for operands that are integers after binary numeric promotion (§ 5.6.2) produces a result value such that  $(a/b) * b + (a\%b)$  is equal to  $a$ .<sup>(1)</sup>

This identity holds even in the special case that the dividend is the negative integer of largest possible magnitude for its type and the divisor is -1 (the remainder is 0).<sup>(2)</sup>

It follows from this rule that the result of the remainder operation can be negative only if the dividend is negative, and can be positive only if the dividend is positive.<sup>(3)</sup>

Moreover, the magnitude of the result is always less than the magnitude of the divisor.<sup>(4)</sup>

If the value of the divisor for an integer remainder operator is 0, then an `ArithmeticException` is thrown.<sup>(5)</sup>

Examples: `5%3` produces 2 (note that  $5/3$  produces 1)  
`5%(-3)` produces 2 (note that  $5/(-3)$  produces -1)  
`(-5)%3` produces -2 (note that  $(-5)/3$  produces -1)  
`(-5)%(-3)` produces -2 (note that  $(-5)/(-3)$  produces 1)<sup>(6)</sup>”

When formalizing the remainder operator, we have to keep in mind the formalization of the division operator and the required equality (1). Therefore, the remainder operator `mod` is formalized as follows:

**defs (overloaded)**

```

mod-def : (x::JavaInt) mod y ≡
  if ((0 < x & y < 0) | (x < 0 & 0 < y)) & ~ (EX z. Rep-JavaInt x = z *
  Rep-JavaInt y)
  then
    Int-to-JavaInt ( Rep-JavaInt x mod Rep-JavaInt y ) - y
  else
    Int-to-JavaInt ( Rep-JavaInt x mod Rep-JavaInt y )

```

For property (5), see the discussion of the division operator above.

Again, it is not made explicit in the JLS what happens if the dividend of a modulo operation equals 0.

Java is not the only language whose definitions of `div` and `mod` do not resemble the mathematical definitions. The languages Fortran, Pascal and Ada define division in the same way as Java, and Fortran’s `MOD` and Ada’s `REM` operators are modelled in the same way as Java’s `%` operator. Goldberg [Gol02, p. H-12] expresses his regret for these differences and argues in favour of the mathematical definition.

## 15.2 Remainder Representation Boundedness

**lemma** *mod-in-JavaInt* [*simp*] : *Rep-JavaInt* *a mod Rep-JavaInt* *b* ∈ *JavaInt*

**proof** –

**have** *Rep-JavaInt* *b* < 0 ⇒ *Rep-JavaInt* *a mod Rep-JavaInt* *b* ∈ *JavaInt*

```

apply (simp add: JavaInt-def)
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
  in IntDiv.neg-mod-bound, simp)
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
  in IntDiv.neg-mod-sign, simp)
apply (rotate-tac)
apply (drule disjI1 [THEN zless-or-eq-imp-zle])
apply (cut-tac b = b in MinInt-leq)
apply (blast intro: order-trans null-leq-MaxInt)
done

```

also have

```

0 < Rep-JavaInt b  $\implies$  Rep-JavaInt a mod Rep-JavaInt b  $\in$  JavaInt
apply (simp add: JavaInt-def)
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
  in pos-mod-bound, simp)
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
  in pos-mod-sign, simp)
apply (cut-tac b = b in MaxInt-geq)
apply (rotate-tac)
apply (drule disjI1 [THEN zless-or-eq-imp-zle])
apply (blast intro: order-trans MinInt-leq0)
done

```

ultimately show *?thesis*

```

apply (cases Rep-JavaInt b < 0)
apply simp
apply (cases Rep-JavaInt b = 0)
apply (simp add: DIVISION-BY-ZERO Rep-JavaInt)
apply (simp add: int-neq-iff)
done

```

qed

**lemma** *mod-minus-divisor-in-JavaInt* [simp] :

**assumes** 1:  $0 < \text{Rep-JavaInt } b$

**shows**  $-\text{Rep-JavaInt } b + \text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b \in \text{JavaInt}$

**proof** –

**have** *HL3a*:  $0 < \text{MaxInt-int}$

**by** (simp add: MaxInt-int-def BitLength-def)

{

**assume** 1:  $0 < \text{Rep-JavaInt } b$

**have**  $\text{MinInt-int} < -\text{Rep-JavaInt } b + \text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b$

**apply** (simp only: zadd-commute[of  $-\text{Rep-JavaInt } b$ ])

**apply** (simp only: zdiff-zless-eq[symmetric, of  $\text{MinInt-int} - -\text{Rep-JavaInt } b$ ])

**apply** (simp )



```

apply (insert MaxInt-geq [of b])
apply (drule zadd-zle-mono1 [of Rep-JavaInt b MaxInt-int MinInt-int])
apply (simp add: MaxInt-MinInt-add)
apply (simp add: zle-add1-eq-le [symmetric, of Rep-JavaInt b + MinInt-int
-1])
apply (insert 1)
apply (drule pos-mod-sign [of Rep-JavaInt b Rep-JavaInt a])
apply simp
done
} note HL1 = this

{
assume 1: 0 < Rep-JavaInt b
have  $- \text{Rep-JavaInt } b + \text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b \leq \text{MaxInt-int}$ 
apply (simp only: zadd-commute[of - Rep-JavaInt b -])
apply (simp only: zdiff-def[symmetric])
apply (simp only: zdiff-zle-eq[of - Rep-JavaInt b MaxInt-int])
apply (insert 1)
apply (drule pos-mod-bound[of Rep-JavaInt b Rep-JavaInt a])
apply (insert HL3a)
apply (drule order-less-imp-le)
apply (drule zadd-zless-mono [of 0 MaxInt-int Rep-JavaInt a mod Rep-JavaInt
b Rep-JavaInt b], simp)
apply (drule order-less-imp-le, simp)
done
} note HL3 = this

show ?thesis
apply (simp add: JavaInt-def)
apply (rule conjI)
apply (insert MinInt-leq0 1)
apply (drule HL1)
apply (simp only: order-less-imp-le)
apply (rule HL3, simp)
done
qed

lemma mod-minus-in-JavaInt :
assumes 2: Rep-JavaInt b ≠ MinInt-int
shows  $\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b - \text{Rep-JavaInt } b \in \text{JavaInt}$ 
proof -

{
assume 1: 0 < Rep-JavaInt b
have  $\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b - \text{Rep-JavaInt } b \in \text{JavaInt}$ 
apply (simp add: JavaInt-def)
apply auto
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
in pos-mod-sign, simp add: 1)

```

```

apply (simp add: zle-zdiff-eq)
apply (subgoal-tac MinInt-int + Rep-JavaInt b ≤ 0)
apply (blast intro: order-trans)

apply (cut-tac b = b in MaxInt-geq)
apply (subgoal-tac MinInt-int + Rep-JavaInt b ≤ MinInt-int + MaxInt-int)
apply (subgoal-tac MinInt-int + MaxInt-int ≤ 0)
apply (blast intro: order-trans)
apply (simp add: MinInt-int-def MaxInt-int-def BitLength-def)
apply (rule zadd-zle-mono2, simp)

apply (simp add: zdiff-zle-eq)
apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
  in pos-mod-bound, simp add: 1)
apply (rotate-tac -1)
apply (drule int-less-le [THEN iffD1])
apply (subgoal-tac Rep-JavaInt b ≤ MaxInt-int + Rep-JavaInt b)
apply (blast intro: order-trans)
apply (simp add: MaxInt-int-def BitLength-def)
done
} note mod-minus-in-JavaInt1 = this

moreover
{
  assume 1: Rep-JavaInt b < 0
  have Rep-JavaInt a mod Rep-JavaInt b - Rep-JavaInt b ∈ JavaInt
    apply (simp add: JavaInt-def)
    apply auto
    apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
      in IntDiv.neg-mod-bound, simp add: 1)
    apply (drule int-less-le [THEN iffD1])
    apply (simp add: zle-zdiff-eq)
    apply auto
    apply (subgoal-tac MinInt-int + Rep-JavaInt b ≤ MinInt-int)
    apply (cut-tac b = b in MinInt-leq)
    apply (blast intro: order-trans)

    apply (simp add: zle-zdiff-eq [symmetric])
    apply (insert 1)
    apply (rotate-tac -1)
    apply (drule int-less-le [THEN iffD1], simp)

    apply (simp add: zdiff-zle-eq)
    apply (cut-tac a = Rep-JavaInt a and b = Rep-JavaInt b
      in IntDiv.neg-mod-sign, simp add: 1)
    apply (subgoal-tac 0 ≤ MaxInt-int + Rep-JavaInt b)
    apply (blast intro: order-trans)
apply (subgoal-tac MaxInt-int + MinInt-int + 1 ≤ MaxInt-int + Rep-JavaInt

```

b)

```
apply (subgoal-tac  $0 \leq \text{MaxInt-int} + \text{MinInt-int} + 1$ )
apply (blast intro: order-trans)
apply (insert 2)
apply (simp add: MaxInt-int-def MinInt-int-def BitLength-def)
apply simp
apply (simp add: add1-zle-eq)
apply (cut-tac  $b = b$  in MinInt-leq)
apply (rotate-tac -1)
apply (drule zle-imp-zless-or-eq)
apply auto
done
} note mod-minus-in-JavaInt2 = this
```

```
moreover
{
  assume 1: Rep-JavaInt  $b = 0$ 
  have Rep-JavaInt  $a \bmod \text{Rep-JavaInt } b - \text{Rep-JavaInt } b \in \text{JavaInt}$ 
    by (simp add: JavaInt-def mod-def DIVISION-BY-ZERO 1)
} note mod-minus-in-JavaInt3 = this
```

```
then show ?thesis
  apply (cases Rep-JavaInt  $b < 0$ )
  apply (simp add: mod-minus-in-JavaInt2)
  apply (simp add: HOL.linorder-not-less)
  apply (drule HOL.order-le-imp-less-or-eq)
  apply auto
  apply (simp add: mod-minus-in-JavaInt1)
done
```

qed

**lemma** *Rep-mod-homom*:

```
assumes 1: ( $0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b$ )
assumes 2:  $\text{EX } z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b$ 
shows  $\text{Rep-JavaInt } (a \bmod b) = \text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b$ 
apply (insert 1 2)
apply (simp add: mod-def split del: split-if)
apply (simp add: Int-to-JavaInt-ident)
done
```

**lemma** *mod-Int-Abs-ident* [simp]:

```
 $\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b) =$   
 $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b)$ 
```

by (rule *Int-to-JavaInt-ident*, simp add: *mod-in-JavaInt*)

**lemma** *div-Int-Abs-ident* :  
assumes *A*: *Rep-JavaInt a*  $\neq$  *MinInt-int* | *Rep-JavaInt b*  $\neq$   $-1$   
shows *Int-to-JavaInt* (*Rep-JavaInt a* div *Rep-JavaInt b*) =  
*Abs-JavaInt* (*Rep-JavaInt a* div *Rep-JavaInt b*)  
apply (rule *Int-to-JavaInt-ident*)  
apply (rule *div-in-JavaInt*, simp)  
apply (insert *A*, simp)  
done

### 15.3 Remainder Characterizations

**lemma** *JavaInt-mod-nondvd-pos-pos*:  
assumes 1: *Rep-JavaInt a* mod *Rep-JavaInt b*  $\neq$  0  
assumes 2:  $0 < a$   
assumes 3:  $0 < b$   
shows  $((a::\text{JavaInt}) \text{ mod } b) = \text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)$   
proof –  
show ?thesis  
apply (insert 1 2 3)  
apply (simp add: *mod-def* split del: *split-if*)  
apply (subst *if-not-P*)  
apply (simp-all add: *Int-to-JavaInt-ident*)  
apply (rule *disjI1*)  
apply (rule *conjI*)  
apply (rule *order-less-not-sym*)  
defer 1  
apply (rule *order-less-not-sym*)  
apply simp-all  
done  
qed

**lemma** *JavaInt-mod-nondvd-neg-neg*:  
assumes 1: *Rep-JavaInt a* mod *Rep-JavaInt b*  $\neq$  0  
assumes 2:  $a < 0$   
assumes 3:  $b < 0$   
shows  $((a::\text{JavaInt}) \text{ mod } b) = \text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)$   
proof –  
show ?thesis  
apply (insert 1,insert 2,insert 3)  
apply (simp add: *mod-def* split del: *split-if*)  
apply (subst *if-not-P*)  
apply (simp-all add: *Int-to-JavaInt-ident*)  
apply (rule *disjI1*)  
apply (rule *conjI*)

```

apply (rule order-less-not-sym)
defer 1
apply (rule order-less-not-sym)
apply simp-all
done
qed

```

```

lemma JavaInt-mod-nondvd-pos-neg:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b  $\neq$  0
  assumes 2: 0 < a
  assumes 3: b < 0
  shows ((a::JavaInt) mod b) = Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt
b) - b
proof -
show ?thesis
  apply (insert 1,insert 2,insert 3)
  apply (simp add:mod-def split del: split-if)
  apply (subst if-P)
  apply (simp-all add: Int-to-JavaInt-ident)

  apply (simp ! add: zmod-eq-0-iff zero-def)
  apply auto
  done
qed

```

```

lemma JavaInt-mod-nondvd-neg-pos:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b  $\neq$  0
  assumes 2: a < 0
  assumes 3: 0 < b
  shows ((a::JavaInt) mod b) = Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt
b) - b
proof -
show ?thesis
  apply (insert 1,insert 2,insert 3)
  apply (simp add:mod-def split del: split-if)
  apply (subst if-P)
  apply (simp-all add: Int-to-JavaInt-ident)

  apply (simp ! add: zmod-eq-0-iff zero-def)
  apply auto
  done
qed

```

## 16 Calculating Example Values for div and mod

### 16.1 Division and Modulo Lemmas: Base Cases

This lemma formalizes the property described in (2) for the division operator (JLS §15.17.2).

Again, the JLS is not very elaborate in (2) regarding the sign of the resulting value if the magnitude of the dividend is less than the magnitude of the divisor. It would have been clearer had they stated the result instead of letting the reader derive the result from the presented inequalities.

**lemma** *MinInt-div-minusone* :

*MinInt div -1 = MinInt*

**proof** –

**have** *MinInt div -1 =*

*Int-to-JavaInt (Rep-JavaInt MinInt div Rep-JavaInt (-1))*

**apply** (*simp add: div-def*)

**apply** (*simp add: number-of-def zero-def*)

**apply** (*subst Int-to-JavaInt-ident [of -1]*)

**apply** (*simp add: minusone-in-JavaInt*)

**apply** (*simp add: JavaInt-less-def JavaInt-le-def*)

**apply** (*simp add: MinInt-def*)

**apply** *auto*

**apply**(*erule allE[of - MinInt-int -]*)

**apply** (*thin-tac ?X*)

**apply** (*thin-tac ?X*) — Isabelle/arith/bug

**apply** *simp*

**done**

**also have** ... =

*Int-to-JavaInt*

*(Rep-JavaInt MinInt div Rep-JavaInt (Int-to-JavaInt (-1)))*

**by** (*simp add: number-of-def*)

**also have** ... =

*Int-to-JavaInt*

*(Rep-JavaInt MinInt div*

*Rep-JavaInt*

*(Abs-JavaInt*

*((-1 + - MinInt-int) mod (2 \* - MinInt-int) + MinInt-int)))*

**by** (*simp only: Int-to-JavaInt-def [of -1]*)

**also have** ... = *Int-to-JavaInt (Rep-JavaInt MinInt div (-1))*

**apply** (*simp add: Rep-Abs-remove*)

**apply** (*simp add: number-of-def MinInt-int-def MaxInt-int-def BitLength-def*)

**done**

**also have** ... = *Int-to-JavaInt (- Rep-JavaInt MinInt)*

**by** *auto*

**also have** ... = *MinInt*

```

    apply (simp add: MinInt-def)

  done

  also from calculation
  show ?thesis by auto
qed

lemma JavaInt-div-dvd:
  assumes 1: Rep-JavaInt a mod Rep-JavaInt b = 0
  shows   ((a::JavaInt) div b) = Int-to-JavaInt(Rep-JavaInt a div Rep-JavaInt
  b)
  proof -
  show ?thesis
    apply (insert 1)
    apply (simp add:div-def split del: split-if)
    apply (subst if-not-P)
    apply (simp ! add: zmod-eq-0-iff zero-def,erule exE)
    apply (rule disjI2)
    apply auto
    done
  qed

lemma JavaInt-0-div [simp]:
  shows 0 div a = (0::JavaInt)
  proof -
  show ?thesis
    apply (subst JavaInt-div-dvd)
    apply (simp-all add: Int-to-JavaInt-ident zero-def)
    done
  qed

lemma JavaInt-div-self [simp]:
  assumes 1: a ≠ 0
  shows   a div a = (1::JavaInt)
  proof -
  show ?thesis
    apply (subst JavaInt-div-dvd)
    apply (simp-all add: one-def 1)
    apply (subst zdiv-self)
    apply (simp-all add: Int-to-JavaInt-ident zero-def 1)
    apply (insert 1)
    apply (erule swap)
    apply (simp add: zero-def Rep-JavaInt-inject[symmetric] )
    done
  qed

```

```

lemma JavaInt-div-by-zero [simp]:
shows  $a \text{ div } 0 = (0::\text{JavaInt})$ 
proof –
show ?thesis
  apply (simp add:div-def zero-def DIVISION-BY-ZERO Int-to-JavaInt-ident
    split del: split-if)
  done
qed

```

```

lemma mod-zero1:
assumes 1:  $\sim a < 0$ 
assumes 2:  $\sim 0 < a$ 
shows  $\text{Rep-JavaInt } a \text{ mod } x = 0$ 
apply (insert 1 2)
apply (drule neg-less-neg-greater-eq, simp)
apply (simp add: zmod-zero zero-def)
done

```

```

lemma mod-zero2:
assumes 1:  $\sim a < 0$ 
assumes 2:  $\sim 0 < a$ 
shows  $x \text{ mod Rep-JavaInt } a = x$ 
apply (insert 1 2)
apply (drule neg-less-neg-greater-eq, simp)
apply (simp add: DIVISION-BY-ZERO zero-def)
done

```

```

lemma mod-zero3:
assumes 1:  $\sim -a < 0$ 
assumes 2:  $\sim 0 < -a$ 
shows  $x \text{ mod Rep-JavaInt } a = x$ 
apply (insert 1 2)
apply (drule neg-less-neg-greater-eq, simp)
apply (drule uminus-zero-zero, simp)
apply (simp add: DIVISION-BY-ZERO zero-def)
done

```

```

lemma mod-zero4:
assumes 1:  $\sim a < 0$ 
assumes 2:  $\sim -a < 0$ 
shows  $x \text{ mod Rep-JavaInt } a = x \ \& \ x \text{ mod Rep-JavaInt } (-a) = x$ 
apply (insert 1 2)
apply (simp add: linorder-not-less)
apply (drule neg-less-zero)
apply (drule leq-geq-eq, simp)
apply (drule uminus-zero-zero, simp)
apply (simp add: DIVISION-BY-ZERO zero-def)
done

```



```

lemma mod-zero5:
  assumes 1:  $\sim 0 < a$ 
  assumes 2:  $\sim 0 < -a$ 
  assumes 3:  $a \neq \text{MinInt}$ 
  shows  $x \bmod \text{Rep-JavaInt } a = x \ \& \ x \bmod \text{Rep-JavaInt } (-a) = x$ 
  apply (insert 1 2 3)
  apply (simp add: linorder-not-less)
  apply (drule pos-neg-eq, simp+)
  apply (simp add: DIVISION-BY-ZERO zero-def)
  done

```

The following three properties show that JavaInt maintains the underlying properties *IntDiv.DIVISION-BY-ZERO*:  $a \text{ div } 0 = 0 \wedge a \text{ mod } 0 = a$ , *IntDiv.zmod-zero*:  $0 \text{ mod } ?b = 0$  and *IntDiv.zmod-self*:  $?a \text{ mod } ?a = 0$  of Integer.

```

lemma DIVISION-BY-ZERO-JMOD [simp]:
  shows  $a \bmod (0::\text{JavaInt}) = a$ 
  by (simp add: mod-def zero-def DIVISION-BY-ZERO Int-to-JavaInt-ident
    Rep-JavaInt
    Rep-JavaInt-inverse)

```

```

lemma mod-self[simp]:
  shows  $(a::\text{JavaInt}) \bmod a = 0$ 
  by (simp add: mod-def Int-to-JavaInt-ident zero-def)

```

```

lemma mod-zero[simp]:
  shows  $0 \bmod (a::\text{JavaInt}) = 0$ 
  apply (simp add: mod-def zero-def)
  done

```

```

lemma JavaInt-mod-0:
  assumes 1:  $\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b = 0$ 
  shows  $((a::\text{JavaInt}) \bmod b) = 0$ 
proof –
show ?thesis
  apply (insert 1)
  apply (simp add: mod-def split del: split-if)
  apply (subst if-not-P)
  apply (simp ! add: zmod-eq-0-iff zero-def,erule exE)
  apply (rule disjI2)
  apply auto
  done

```

**qed**

In Java, the expression  $c = \text{MinInt} \bmod a$  gives quite chaotic results. For the special case  $a = 2^k$  for some  $k$ , it can be established that  $c$  is zero. For all other cases, we we have been unable to find any pattern.

```

lemma mod-eq-0-iff2 :

```

```

assumes 1: EX z. Rep-JavaInt x = z * Rep-JavaInt y
shows  $x \bmod y = 0$ 
apply (rule JavaInt-mod-0)
apply (subst zmod-eq-0-iff)
apply (simp add: 1 zmult-commute)
done

```

**lemma** *neg-charn1*:

```

assumes 1: neg (Rep-JavaInt n) = neg (Rep-JavaInt d)
shows (Rep-JavaInt n < 0 & Rep-JavaInt d < 0) |
(0 ≤ Rep-JavaInt n & 0 ≤ Rep-JavaInt d)
apply (insert 1)
apply (simp add: neg-eq-less-0)
apply auto
done

```

**lemma** *neg-charn2*:

```

assumes 1: neg (Rep-JavaInt n) ≠ neg (Rep-JavaInt d)
shows (Rep-JavaInt n < 0 & 0 ≤ Rep-JavaInt d) |
(0 ≤ Rep-JavaInt n & Rep-JavaInt d < 0)
apply (insert 1)
apply (simp add: neg-eq-less-0)
apply auto
done

```

**lemma** *Factor-in-JavaInt* :

```

assumes 1: Rep-JavaInt a = z * Rep-JavaInt b
assumes 2: Rep-JavaInt b ≠ 0
assumes 3: Rep-JavaInt a ≠ MinInt-int | Rep-JavaInt b ≠ -1
shows  $z \in \text{JavaInt}$ 
proof –
{
  assume A: Rep-JavaInt a = MinInt-int
  assume A2: Rep-JavaInt b ≠ -1
  have  $z \in \text{JavaInt}$ 
  apply (insert 1 2)
  apply (drule divide-both-sides [of Rep-JavaInt a z * Rep-JavaInt b Rep-JavaInt
b])
  apply (drule zdiv-zmult-self1 [of Rep-JavaInt b z])
  apply simp
  apply (hypsubst)
  apply (rule div-in-JavaInt [of a b])
  apply (simp add: A A2)
  done
} note factor-minint = this

```

**also**

```
{
```

```

assume B1: Rep-JavaInt a = MinInt-int
assume B2: Rep-JavaInt b = -1
have z ∈ JavaInt
  apply (insert 3 B1 B2)
  apply simp
  done
} note factor-minint-minusone = this

moreover
{
  assume B: Rep-JavaInt a ≠ MinInt-int
  assume C: Rep-JavaInt b ≠ 0
  have z ∈ JavaInt
    apply (insert 1 C)
  apply (drule divide-both-sides [of Rep-JavaInt a z * Rep-JavaInt b Rep-JavaInt
b])
    apply (drule zdiv-zmult-self1 [of Rep-JavaInt b z])
    apply simp
    apply (hypsubst)
    apply (rule div-in-JavaInt [of a b])
    apply (simp add: B)
    done
  } note factor-nothing = this

moreover
{
  assume D: Rep-JavaInt b = 0
  have z ∈ JavaInt
    apply (insert 2 D)
    apply simp
    done
  } note factor-null = this

show ?thesis
  apply (insert 1)
  apply (cases Rep-JavaInt a = MinInt-int)
  apply (cases Rep-JavaInt b = -1)
  apply (simp add: factor-minint-minusone)
  apply (simp-all add: factor-minint)
  — now: Rep-JavaInt a ≠ MinInt-int
  apply (cases Rep-JavaInt b ≠ 0)
  apply (simp-all add: factor-nothing)
  apply (simp-all add: factor-null)
  done
qed

```

These lemmas formalize the property described in (2) for the modulo operator (JLS §15.17.3).

**lemma** *MinInt-mod-minusone[simp]*:

```

MinInt mod -1 = 0
  apply (simp add: mod-def)
  apply auto
  apply (simp add: JavaInt-less-def JavaInt-le-def number-of-def zero-def)
  apply (simp-all add: MinInt-def number-of-def zero-def)
done

```

## 16.2 Sign of Divisions

```

lemma JavaInt-div-neg-shift:
  assumes 1:a ≠ MinInt
  assumes 2:b ≠ MinInt
  shows   a div (-b) = (-a) div b
proof -
{assume 1: a ≠ MinInt
have     (Rep-JavaInt a mod -(Rep-JavaInt b) = 0) =
          (Rep-JavaInt a mod Rep-JavaInt b = 0)
  apply (cases Rep-JavaInt a mod Rep-JavaInt b = 0)
  apply (subst zmod-zminus2-eq-if)
  apply simp

  apply simp
  apply (subst zmod-zminus2-eq-if)
  apply simp

  apply (cases 0 < Rep-JavaInt b)
  apply (drule IntDiv.pos-mod-bound [of Rep-JavaInt b Rep-JavaInt a])
  apply simp

  apply (simp only: linorder-not-less)
  apply (drule HOL.order-le-imp-less-or-eq)
  apply (cases Rep-JavaInt b < 0)
  apply (drule IntDiv.neg-mod-bound [of Rep-JavaInt b Rep-JavaInt a])
  apply simp

  apply simp
  done
} note HL2 = this

{assume 1: a ≠ MinInt
have     (-(Rep-JavaInt a) mod Rep-JavaInt b = 0) =
          (Rep-JavaInt a mod Rep-JavaInt b = 0)
  apply (insert 1)
  apply (drule HL2)
  apply (simp add: zmod-zminus2 [of Rep-JavaInt a Rep-JavaInt b])
  done
} note HL3 = this

show ?thesis

```

```

apply(cases Rep-JavaInt a mod Rep-JavaInt b = 0)
apply(simp add: 1 2 uminus-homom HL2 HL3 JavaInt-div-dvd zdiv-zminus2 )

apply(cases a < 0)
apply(cases b < 0)
prefer 3
apply(cases b < 0)

apply(simp-all add: linorder-not-less order-le-less)
apply auto

apply(subst JavaInt-div-nondvd-neg-neg)
apply(simp-all add: JavaInt-div-nondvd-neg-neg JavaInt-div-nondvd-neg-pos
  JavaInt-div-nondvd-pos-neg JavaInt-div-nondvd-pos-pos
  uminus-pos-if-neg uminus-neg-if-pos 1 2 uminus-homom
  HL2 HL3
  zdiv-zminus2)

done
qed

```

### 16.3 Sign of Remainders

The following two lemmas relate the signs of the modulo operands. Interestingly enough, they differ quite a lot from the corresponding lemmas on the *int* level, which are

*IntDiv.zmod-zminus-zminus*:  $- ?a \text{ mod } - ?b = - (?a \text{ mod } ?b)$  and  
*IntDiv.zmod-zminus2*:  $?a \text{ mod } - ?b = - (- ?a \text{ mod } ?b)$

```

lemma JavaInt-mod-neg1:
  assumes 1:a ≠ MinInt
  assumes 2:b ≠ MinInt
  shows -((a::JavaInt) mod b) = (-a) mod b
proof -
{assume 1: a ≠ MinInt
have (- (Rep-JavaInt a) mod Rep-JavaInt b = 0) =
  (Rep-JavaInt a mod Rep-JavaInt b = 0)
  apply (simp add: zmod-zminus1-eq-if)
  apply (rule impI)
  apply (simp add: zdiff-eq-eq [of Rep-JavaInt b - 0])
  apply (cases 0 < Rep-JavaInt b)
  apply (drule IntDiv.pos-mod-bound [of Rep-JavaInt b Rep-JavaInt a])
  apply (simp add: HOL.neq-commute [of Rep-JavaInt b -]
    HOL.order-less-le)

  — 2nd case:
  apply (simp add: linorder-not-less)
  apply (cases Rep-JavaInt b = 0)
  apply (simp add: DIVISION-BY-ZERO)

```

```

— 3rd case:
apply (drule conjI [of Rep-JavaInt b ≤ 0 Rep-JavaInt b ≠ 0], simp)
apply (simp add: HOL.order-less-le[symmetric])
apply (drule IntDiv.neg-mod-bound [of Rep-JavaInt b Rep-JavaInt a])
apply simp
done
} note HL2 = this
{have Rep-JavaInt b ≠ MinInt-int
  apply (insert 2)
  apply (simp add: Rep-JavaInt-inject[symmetric, of b MinInt])
  apply (simp add: MinInt-def Abs-JavaInt-inverse)
  done
} note HL3 = this
{have Rep-JavaInt a ≠ MinInt-int
  apply (insert 1)
  apply (simp add: Rep-JavaInt-inject[symmetric, of a MinInt])
  apply (simp add: MinInt-def Abs-JavaInt-inverse)
  done
} note HL3a = this

{have MinInt-int + Rep-JavaInt b < 0
  apply (simp only: zless-zdiff-eq[symmetric,
    of MinInt-int Rep-JavaInt b 0])
  apply simp
  done
} note HL3b = this

{have Rep-JavaInt b - Rep-JavaInt a mod Rep-JavaInt b ∈ JavaInt
  apply (insert HL3)
  apply (drule mod-minus-in-JavaInt[of b a])
  apply (simp only: zminus-zdiff-eq[symmetric, of Rep-JavaInt b -])
  apply (rule minus-in-JavaInt2 [of
    (Rep-JavaInt a mod Rep-JavaInt b) - Rep-JavaInt b], simp)
  apply (simp add: zdiff-eq-eq)
  apply (cases 0 < Rep-JavaInt b)
  apply (frule IntDiv.pos-mod-sign [of Rep-JavaInt b Rep-JavaInt a])
  apply (insert MaxInt-MinInt-add)
  apply (insert Rep-JavaInt [of b])
  apply (drule MaxInt-int-is-largest [of Rep-JavaInt b])
  apply (insert HL3b)
  apply (drule order-less-le-trans [of MinInt-int + Rep-JavaInt b 0
    Rep-JavaInt a mod Rep-JavaInt b], simp)
  apply simp — finished

— now prove the last subgoal:
apply (simp add: linorder-not-less)
apply (cases Rep-JavaInt b = 0)
apply (simp add: DIVISION-BY-ZERO)
apply (insert HL3a, simp)

```

```

apply (drule conjI [of Rep-JavaInt  $b \leq 0$  Rep-JavaInt  $b \neq 0$ ], simp)
apply (simp add: order-less-le[symmetric])
apply (drule IntDiv.neg-mod-bound [of Rep-JavaInt  $b$  Rep-JavaInt  $a$ ])
apply (insert MinInt-less0)
apply (drule zless-not-sym [of MinInt-int 0])
apply auto
done
} note HL4 = this

show ?thesis
apply (cases Rep-JavaInt  $a$  mod Rep-JavaInt  $b = 0$ )
  apply (simp add: uminus-homom HL2 1 JavaInt-mod-0)
  — case Rep-JavaInt  $a$  mod Rep-JavaInt  $b \neq 0$ 
apply (cases  $a < 0$ )
apply (cases  $b < 0$ )
prefer 3
apply (cases  $b < 0$ )

apply (simp-all add: linorder-not-less order-le-less)
apply auto
apply (simp-all add: JavaInt-mod-nondvd-neg-neg JavaInt-mod-nondvd-neg-pos
  JavaInt-mod-nondvd-pos-neg JavaInt-mod-nondvd-pos-pos)

apply (insert 1)
apply (drule uminus-neg-if-pos) defer
apply (drule uminus-neg-if-pos) defer
apply (drule uminus-pos-if-neg, simp) defer
apply (drule uminus-pos-if-neg, simp) defer
apply (insert 1)
apply (simp-all add: HL2[symmetric])
apply (simp-all add: uminus-homom[symmetric])
apply (simp-all add: JavaInt-mod-nondvd-neg-neg JavaInt-mod-nondvd-neg-pos
  JavaInt-mod-nondvd-pos-neg JavaInt-mod-nondvd-pos-pos)
apply (simp-all add: uminus-homom)
apply (simp-all add: zmod-zminus1-eq-if)
apply auto
apply (simp-all only: JavaInt-add-commute [of  $- b$  -])
apply (simp-all only: Ring.ring.minus-def[symmetric])
apply (simp-all add: diff-def)
apply (simp-all add: Int-to-JavaInt-ident HL4)
apply (simp-all add: uminus-def)
done
qed

lemma JavaInt-mod-neg2:
  shows  $a \bmod (-b) = (a::JavaInt) \bmod b$ 
proof —
  {

```

```

    fix x
    have (∀ y. (x::int) ≠ - y) ⇒ False
      by (drule spec [of - - x], auto)
  } note H1-1 = this

also
have H1 : EX q. Rep-JavaInt a = q * Rep-JavaInt b ⇒
  EX q. Rep-JavaInt a = - q * Rep-JavaInt b
  apply auto
  apply (simp only: zmult-zminus[symmetric])
  apply (simp only: zmult-cancel2)
  apply auto
  apply (drule H1-1)
  apply auto
  done

moreover
{
  fix z b
  have ∀ q. - (z * Rep-JavaInt b) ≠ q * Rep-JavaInt b ⇒
    False
    by (drule spec [of - - z], auto)
} note H2 = this

{
  assume X1: b = MinInt
  have a mod (-b) = (a::JavaInt) mod b
    apply (insert X1)
    apply auto
    apply (simp add: uminus-MinInt)
  done
} note case-MinInt = this

{ fix a b
  assume X1: b ≠ MinInt
  have Rep-JavaInt a mod Rep-JavaInt (- b) = Rep-JavaInt a mod - Rep-JavaInt
  b
    apply (simp add: uminus-def)
    apply (subst Int-to-JavaInt-ident [of - Rep-JavaInt b])
    defer 1
    apply (subst Abs-JavaInt-inverse)
    defer 1
    apply simp
    apply (tactic distinct-subgoals-tac)
    apply (insert X1)
    apply (simp add: Rep-JavaInt-inject[symmetric] MinInt-def)
  done
} note reshape = this

```



```

{
  fix a b
  assume X1:  $\forall z. \text{Rep-JavaInt } a \neq z * \text{Rep-JavaInt } b$ 
  assume X2:  $b \neq \text{MinInt}$ 

  have
    Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)) =
      - b + Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)
    apply (subst reshape, simp add: X2)
    apply (simp add: add-def uminus-def)
    apply (subst Int-to-JavaInt-ident [of - Rep-JavaInt b])
    defer 1
    apply (subst Abs-JavaInt-inverse [of - Rep-JavaInt b])
    defer 1
    apply (subst zmod-zminus2-eq-if)
    apply (cases Rep-JavaInt a mod Rep-JavaInt b = 0)
    apply simp
    defer 1
    apply (subst if-not-P, simp)
    apply (simp only: zadd-commute [of - Rep-JavaInt b Rep-JavaInt a mod
Rep-JavaInt b]
      zdiff-def [symmetric])

    apply (tactic distinct-subgoals-tac)
    defer 1

    apply (insert X1)
    apply auto

    apply (insert X2)
    apply (simp add: Rep-JavaInt-inject[symmetric] MinInt-def)
    apply (subst Int-to-JavaInt-ident)
    apply auto
    apply (rule mod-minus-in-JavaInt)
    apply (simp add: not-MinInt-not-MinInt-int)
    done
} note L1 = this

{ fix a b q
  assume 1:  $\text{Rep-JavaInt } a = q * \text{Rep-JavaInt } (- b)$ 
  assume 2:  $b \neq \text{MinInt}$ 
  have  $(q * \text{Rep-JavaInt } (- b)) \text{ mod Rep-JavaInt } b = 0$ 
    apply (simp add: uminus-def times-def)
    apply (insert 2)
    apply (subst Int-to-JavaInt-ident)
    defer 1
    apply (subst Abs-JavaInt-inverse)
    defer 1
    apply (simp only: mult-uminus-shift)

```

```

apply (simp only: zmod-zmult-self1)
apply (tactic distinct-subgoals-tac)
apply (simp add: not-MinInt-not-MinInt-int)
apply (simp add: minus-in-JavaInt)
done
} note L2H1 = this

{
assume X1: b ≠ MinInt
assume X2: (0 < a → ~ b < 0) & (a < 0 → ~ 0 < b) |
  (EX z. Rep-JavaInt a = z * Rep-JavaInt b)
assume X3: (0 < a → ~ - b < 0) & (a < 0 → ~ 0 < - b) |
  (EX z. Rep-JavaInt a = z * Rep-JavaInt (- b))

have Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)) =
  Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)

apply (insert X2)
apply (cases (0 < a → ~ b < 0) & (a < 0 → ~ 0 < b))
apply (thin-tac ?X | ?Y)
defer 1
apply simp
apply (drule Rep-mod-homom, simp)
apply (drule mod-eq-0-iff2)

apply (insert X3)
apply (cases (0 < a → ~ - b < 0) & (a < 0 → ~ 0 < - b))
apply (thin-tac ?X | ?Y)
defer 1
apply simp
apply (drule Rep-mod-homom, simp)
apply (drule mod-eq-0-iff2)
apply simp

apply auto

apply (simp-all add: mod-zero1)
apply (simp-all add: mod-zero2)
apply (simp-all add: mod-zero3)
apply (simp add: mod-zero5 X1)
apply (simp add: mod-zero4)

apply (subst L2H1 [of a - b], (simp add: X1)+) +

apply (drule neg-less-neg-greater-eq [of b 0], simp)
apply (simp add: uminus-def)
apply (simp add: Int-to-JavaInt-ident minus-in-JavaInt zero-def)

apply (subst reshape, simp add: X1)

```

```

apply (simp add: zero-def)
apply (simp add: zmod-zminus2-eq-if zero-def)

apply (subst reshape, simp add: X1)
apply (simp add: zero-def)
apply (simp add: zmod-zminus2-eq-if zero-def)
done
} note L2 = this

have stupid:
(Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b) =
  b + Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)))
=
(Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- (- b))) =
  - (- b) + Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)))
by (simp add: ring-simps)

{
assume X1: b ≠ MinInt
assume X2: ∀ z. Rep-JavaInt a ≠ z * Rep-JavaInt (- b)

have
  b + Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)) =
    Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)
apply (simp only: HOL.eq-commute [of
  b + Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt (- b)) -])
apply (subst stupid)
apply (rule L1)
apply assumption
apply (insert X1)
apply (erule swap)
apply simp
apply (drule eq-times-eq [of - b MinInt -1])
apply simp
apply (simp only: MinInt-times-minusone)
apply (simp only: uminus-times-minusone)
apply (simp add: ring-simps)
done
} note L3 = this

{
assume X1: b ≠ MinInt
assume X2: (0 < a & b < 0 | a < 0 & 0 < b)
assume X3: (0 < a & - b < 0 | a < 0 & 0 < - b)

have
  False
apply (insert X2 X3)
apply (drule disjE [of - - False])

```

```

apply simp-all
apply (drule disjE [of - - False])
prefer 4
apply (drule disjE [of - - False])
apply simp-all
apply (drule conjE [of - - False], simp-all) +
defer
apply (drule conjE [of - - False], simp-all) +
defer
apply (drule conjE [of - - False], simp-all) +
defer
apply (drule conjE [of - - False], simp-all) +
apply (insert X1)
defer
apply (drule uminus-neg1, simp)
apply (drule less-greater-contradict, simp+)
apply (drule uminus-neg2, simp)
apply (drule less-greater-contradict, simp+)
apply (drule uminus-neg1, simp)
apply (drule less-greater-contradict, simp+)
apply (drule less-greater-contradict, simp+)
done
} note  $L_4 = \text{this}$ 

show ?thesis

apply (cases b = MinInt)
apply (rule case-MinInt, simp)

apply (subst mod-def)
apply (cases ( $0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b$ )
  &  $\sim$  ( $\text{EX } z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b$ ))
apply simp — positive part is solved by simp
defer 1
apply (subst if-not-P, simp) — negative part needs some more help

apply (simp)
apply (subst mod-def)
apply (cases ( $0 < a \ \& \ - b < 0 \mid a < 0 \ \& \ 0 < - b$ ) &
   $\sim$  ( $\text{EX } z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } (- b)$ ))
apply simp — positive part is solved by simp
defer 1
apply (subst if-not-P, simp) — negative part needs some more help

prefer 2
apply (subst mod-def)
apply (cases ( $0 < a \ \& \ - b < 0 \mid a < 0 \ \& \ 0 < - b$ ) &
   $\sim$  ( $\text{EX } z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } (- b)$ ))
apply simp — positive part is solved by simp

```

```

defer 1
apply (subst if-not-P, simp) — negative part needs some more help
apply simp
apply (rule L1, simp)
apply (simp, simp)

apply (rule L2, simp)
apply (simp, simp)

apply (rule L3, simp)
apply simp

apply (drule L4)
apply (simp, simp, simp)
done
qed

```

## 16.4 Bounds of the Division Operation

**lemma** *abs-explain*:

```

assumes A1:  $abs\ (Rep\ JavaInt\ d) \leq abs\ (Rep\ JavaInt\ n)$ 
assumes A2:  $neg\ (Rep\ JavaInt\ d) = neg\ (Rep\ JavaInt\ n)$ 
shows  $(Rep\ JavaInt\ d \leq Rep\ JavaInt\ n \ \&\ 0 \leq Rep\ JavaInt\ n)$ 
  |  $(Rep\ JavaInt\ n \leq Rep\ JavaInt\ d \ \&\ Rep\ JavaInt\ n \leq 0)$ 
apply (insert A1 A2)
apply (drule neg-cha1)
apply (simp add: zabs-def)
apply auto
done

```

**lemma** *abs-explain-neg*:

```

assumes A1:  $abs\ (Rep\ JavaInt\ d) \leq abs\ (Rep\ JavaInt\ n)$ 
assumes A2:  $neg\ (Rep\ JavaInt\ d) \neq neg\ (Rep\ JavaInt\ n)$ 
shows  $(- Rep\ JavaInt\ d \leq Rep\ JavaInt\ n \ \&\ 0 \leq Rep\ JavaInt\ n)$ 
  |  $(Rep\ JavaInt\ n \leq - Rep\ JavaInt\ d \ \&\ Rep\ JavaInt\ n \leq 0)$ 
apply (insert A1 A2)
apply (drule neg-cha2)
apply (simp add: zabs-def)
apply auto
done

```

This lemma formalizes the first property described in (1) for the division operator (JLS §15.17.2).

The predicate “neg” holds iff the value of its argument is less than zero.

Again, the phrasing in the JLS is quite imprecise as the “one special case that does not satisfy this rule” refers (in the text) to the lemma *quotient-sign-plus* as well as to the sister lemma *quotient-sign-minus* (see below). In fact, the lemma *quotient-sign-minus* is not touched by the special case  $n = MinInt$

and  $d = (-1::'a)$  because  $n$  and  $d$  have opposite signs. Therefore, we exclude the special case only in the lemma *quotient-sign-plus*.

In our total formalization, we also have to exclude the special case  $d = (0::'a)$  because  $a \text{ div } (0::'a) = (0::'a)$  holds in our total world, which would contradict the lemma. Therefore, for  $d = (0::'a)$  we cannot prove  $(0::'a) < a \text{ div } (0::'a)$ . This does not really hurt as  $d = (0::'a)$  is excluded in the exception layer anyways.

The JLS restricts the lemmas *quotient-sign-plus* and *quotient-sign-minus* to the cases where  $|n| \geq |d|$  without specifying what happens for the other cases. Test runs showed that in Java, we always have  $n \text{ div } d = (0::'a)$  if  $|d| > |n|$ . This fact is missing in the JLS. Additionally, it could be checked here as well.

**lemma** *quotient-sign-plus* :

**assumes** 1:  $(\text{abs } (\text{Rep-JavaInt } d) \leq \text{abs } (\text{Rep-JavaInt } n))$

— we have to regard the int values, no wraparound!

**assumes** 2:  $\text{neg } (\text{Rep-JavaInt } n) = \text{neg } (\text{Rep-JavaInt } d)$

**assumes** 3:  $n \neq \text{MinInt} \mid d \neq -1$

**assumes** 4:  $d \neq 0$

**shows**  $0 < (n \text{ div } d)$

**proof** —

**have** *HL4* [*simp*] :  $\text{Rep-JavaInt } n \neq \text{MinInt-int} \mid \text{Rep-JavaInt } d \neq (-1::\text{int})$

**apply** (*insert 3*)

**apply** (*subst Abs-JavaInt-inject[symmetric, of Rep-JavaInt n MinInt-int]* )

**apply** (*simp-all add: Rep-JavaInt*)

**apply** (*subst Abs-JavaInt-inject[symmetric, of Rep-JavaInt d -1]*)

**apply** (*simp-all add: Rep-JavaInt*)

**apply** (*simp add: Rep-JavaInt-inverse MinInt-def[symmetric]* )

**done**

**have** *HL3* [*simp*] :  $\text{Rep-JavaInt } n \text{ div } \text{Rep-JavaInt } d \in \text{JavaInt}$

**apply** (*rule div-in-JavaInt [of n d]*)

**apply** (*simp only: HL4*)

**done**

**moreover** {

**fix**  $n \ d$

**assume** *TL1-1*:  $n = \text{MinInt}$

**have**  $(\text{Rep-JavaInt } n) \leq (\text{Rep-JavaInt } d)$

**apply** (*insert TL1-1 MinInt-is-least2[of d]*)

**apply** (*simp add: JavaInt-le-def*)

**done**

} **note** *TL1* = *this*

{

**assume** *A1*:  $0 \leq \text{Rep-JavaInt } n$

**have**  $0 < \text{Rep-JavaInt } n$

**apply** (*insert 1 A1 4*)

```

    apply (simp add: Rep-JavaInt-inject[symmetric] zero-def)
    apply (drule abs-neq0-greater-neq0, assumption)
    apply (simp add: zabs-def)
  done
} note ineq2 = this

{
  assume A1: 0 ≤ Rep-JavaInt d
  have 0 < Rep-JavaInt d
    apply (insert 4 A1)
    apply (simp add: Rep-JavaInt-inject[symmetric] zero-def)
  done
} note ineq3 = this

{
  assume TL2-1: (Rep-JavaInt n < 0 & Rep-JavaInt d < 0) | (0 ≤ Rep-JavaInt
n & 0 ≤ Rep-JavaInt d)
  have n < 0 & d < 0 | 0 ≤ n & 0 ≤ d
    apply (insert TL2-1)
  apply (simp add: JavaInt-less-def JavaInt-le-def zero-def
Rep-JavaInt-inject[symmetric])
  apply (simp only: int-less-le[of Rep-JavaInt d 0])
  apply (simp only: int-less-le[of Rep-JavaInt n 0])
  apply simp
  done
} note TL2 = this

{
  have n < 0 & d < 0 | 0 ≤ n & 0 ≤ d → ∼(0 < n & d < 0 | n < 0 & 0 <
d)
  apply auto
  apply (simp add: linorder-not-le[symmetric, of n 0] order-less-imp-le)
  apply (simp add: linorder-not-le[symmetric, of d 0] order-less-imp-le)
  apply (simp add: linorder-not-le[symmetric, of d 0])
  apply (simp add: linorder-not-le[symmetric, of n 0])
  done
} note TL3 = this

{
  assume Absch2-1: Rep-JavaInt n < 0 & Rep-JavaInt d < 0
  have Rep-JavaInt n ≤ Rep-JavaInt d
    apply (insert 1 2)
  apply (drule abs-explain, simp)
  apply (insert Absch2-1)
  apply auto
  done
} note Absch2 = this

```

```

{ assume Absch3-1: Rep-JavaInt  $n < 0 \longrightarrow \sim \text{Rep-JavaInt } d < 0$ 
  have  $0 < \text{Rep-JavaInt } n$ 
    apply (insert 2 Absch3-1)
    apply (drule neg-charn1)
    apply auto
    apply (simp-all add: linorder-not-less)
    apply (drule ineq2, simp)+
    done
} note Absch3 = this

{ assume Absch4-1: Rep-JavaInt  $n < 0 \longrightarrow \sim \text{Rep-JavaInt } d < 0$ 
  have  $0 < \text{Rep-JavaInt } d$ 
    apply (insert 2 Absch4-1)
    apply (drule neg-charn1)
    apply auto
    apply (simp-all add: linorder-not-less)
    apply (drule ineq3, simp)+
    done
} note Absch4 = this

{
  assume Absch1-1: Rep-JavaInt  $n < 0 \longrightarrow \sim \text{Rep-JavaInt } d < 0$ 
  have Rep-JavaInt  $d \leq \text{Rep-JavaInt } n$ 
    apply (insert Absch1-1)
    apply (frule Absch3)
    apply (drule Absch4)
    apply (insert 1)
  apply (drule abs-explain, simp add: 2)
  apply auto
  done
} note Absch1 = this

show ?thesis

apply (simp add: div-def split del: split-if)

apply (insert 2)
apply (drule neg-charn1)

apply (drule TL2)

apply (insert TL3)

apply (drule mp [of  $n < 0 \ \& \ d < 0 \mid 0 \leq n \ \& \ 0 \leq d$  -], assumption)
apply (simp add: if-not-P if-False)
apply (thin-tac ?X)+
  — remove all assumptions

```



— remove Int-to-JavaInt-ident:  
**apply** (*subst Int-to-JavaInt-ident, simp*)  
**apply** (*simp add: JavaInt-less-def JavaInt-le-def*  
*Rep-JavaInt-inject[symmetric] zero-def*)  
**apply** (*simp only: int-less-le[symmetric] HOL.eq-commute*)

**apply** (*cases Rep-JavaInt n < 0 & Rep-JavaInt d < 0*)  
**apply** (*rule div-neg-neg-gt0*)  
**prefer** 4  
**apply** (*rule div-pos-pos-gt0*)

**apply** *simp-all*

**apply** (*drule Absch1, simp*)  
**apply** (*drule Absch3, simp*)  
**apply** (*drule Absch4, simp*)  
**apply** (*drule Absch2, simp*)  
**done**

**qed**

**lemma** *null-less-MaxInt [simp]: 0 < MaxInt-int*  
**by** (*simp add: MaxInt-int-def BitLength-def*)

**lemma** *quotient-sign-minus1 :*  
**assumes** 1: *Rep-JavaInt d < 0*  
**assumes** 2: *0 < Rep-JavaInt n*  
**assumes** 3: *- Rep-JavaInt d ≤ Rep-JavaInt n*  
**shows**  $(n \text{ div } d) < 0$

**proof** —

**have** *HL4 [simp]: Rep-JavaInt n ≠ MinInt-int | Rep-JavaInt d ≠ (-1::int)*  
**apply** (*insert 1 2*)  
**apply** *auto*  
**apply** (*insert MinInt-less0*)  
**apply** *simp*  
**done**

**have** *HL3 [simp]: Rep-JavaInt n div Rep-JavaInt d ∈ JavaInt*  
**apply** (*rule div-in-JavaInt [of n d]*)  
**apply** (*simp only: HL4*)  
**done**

```

moreover{
  fix  $x\ y$ 
  have  $\llbracket y \leq x; x \neq y \rrbracket \implies y < (x::int)$ 
    by (simp only: HOL.neg-commute[ $of\ x\ y$ ])
} note  $HL5 = this$ 

moreover
have  $HL2 : \llbracket Rep\text{-}JavaInt\ n < 0; 0 \leq Rep\text{-}JavaInt\ d \rrbracket$ 
   $\implies Rep\text{-}JavaInt\ n\ div\ Rep\text{-}JavaInt\ d + 1 \in JavaInt$ 
  apply (simp add: JavaInt-def, auto)
  apply (insert HL4)
  apply (drule div-in-JavaInt [ $of\ n\ d$ ])
  apply (simp add: JavaInt-def)
  apply auto

  apply (cases Rep-JavaInt d = 0)
  apply (simp !)
  apply (drule HL5[ $of\ 0\ Rep\text{-}JavaInt\ d$ ], simp)
  apply (drule div-neg-pos-less0 [ $of\ Rep\text{-}JavaInt\ n\ Rep\text{-}JavaInt\ d$ ], simp)
  apply (simp add: add1-zle-eq)
  apply (insert null-leq-MaxInt)
  apply (drule order-less-le-trans [ $of\ -\ 0\ MaxInt\text{-}int$ ])
  apply simp-all
done

moreover
have  $HL2b : \llbracket 0 \leq Rep\text{-}JavaInt\ n; Rep\text{-}JavaInt\ d < 0 \rrbracket$ 
   $\implies Rep\text{-}JavaInt\ n\ div\ Rep\text{-}JavaInt\ d + 1 \in JavaInt$ 
  apply (simp add: JavaInt-def, auto)
  apply (insert HL4)
  apply (drule div-in-JavaInt [ $of\ n\ d$ ])
  apply (simp add: JavaInt-def)
  apply (auto)
  — remove duplicate subgoals:
  apply (thin-tac [!]  $?X \neq ?Y$ )
  apply (tactic distinct-subgoals-tac)
  — prove the remaining subgoal:
  apply (drule div-nonneg-neg-le0, simp)
  apply (simp add: add1-zle-eq)
  apply (insert null-less-MaxInt)
  apply (drule order-le-less-trans [ $of\ -\ 0\ MaxInt\text{-}int$ ])
  apply simp-all
done

moreover {
  fix  $x ::int$ 
  have  $EX\ z. -\ Rep\text{-}JavaInt\ d - Rep\text{-}JavaInt\ d * x = z * Rep\text{-}JavaInt\ d$ 
    apply (rule exI [ $of\ -\ -\ 1 - x$ ])
    apply (simp add: zdiff-zmult-distrib)

```

```

    apply (simp add: zmult-commute)
  done
} note HL1 = this

show ?thesis
  apply (simp add: div-def split del: split-if)
  apply (insert 1 2 3)
  — case distinction: rounding towards zero or not
  apply (cases (0 < n & d < 0 | n < 0 & 0 < d) &
    ~ (EX z. Rep-JavaInt n = z * Rep-JavaInt d))
  apply (simp-all add:if-not-P split del: split-if)
  apply (simp only: JavaInt-add-commute
    [of 1 Int-to-JavaInt (Rep-JavaInt n div Rep-JavaInt d)])
  apply (simp-all add: Int-to-JavaInt-ident Int-to-JavaInt-homom HL2b
    JavaIntegersAdd.add-def zero-def one-def le-quasi-def)
  apply (simp-all add: neg-imp-zdiv-neg-iff, simp ! add:order-le-less)

  — remaining case: rounding.
  — Let  $- \text{Rep-JavaInt } d = \text{Rep-JavaInt } n$ 
  apply (erule disjE, simp add: zless-iff-Suc-zadd[of - Rep-JavaInt d -])
  prefer 2
  — then dividable, no rounding should happen, contradiction
  apply(erule allE[of - -1 -])
  apply auto

  — Case:  $- \text{Rep-JavaInt } d < \text{Rep-JavaInt } n$ . Then  $\text{Rep-JavaInt } n = - \text{Rep-JavaInt } d + (1 + \text{int } na)$ .
  apply(subst zminus-zminus[symmetric, of Rep-JavaInt d])
  apply(subst zdiv-zminus2-eq-if, simp ! add: order-less-le)

  apply(subgoal-tac (- Rep-JavaInt d +(1+int na)) mod - Rep-JavaInt d ≠ 0)
  prefer 2
  apply (simp only: not-ex[symmetric],erule swap)
  apply (simp ! add: zmod-eq-0-iff,erule exE,simp add: HL1)

  apply (simp add: zdiff-zless-eq)
  apply (rule order-less-le-trans[of - 0 -], simp)
  apply (subst pos-imp-zdiv-nonneg-iff, simp,simp)

done
qed

```

```

lemma quotient-sign-minus2 :
  assumes 1: 0 < Rep-JavaInt d
  assumes 2: Rep-JavaInt n < 0
  assumes 3: Rep-JavaInt n ≠ MinInt-int
  assumes 4: Rep-JavaInt n ≤ - Rep-JavaInt d

```

```

shows      (n div d) < 0
proof –

have HL1: d ≠ MinInt
  apply (insert 1)
  apply (simp add: greater-zero-Abs-Rep[symmetric])
  apply (drule larger-zero-neq-MinInt [of d])
  apply (simp add: neq-commute)
done

show ?thesis
  apply (insert 3)
  apply (simp only: not-MinInt-not-MinInt-int[symmetric])
  apply (insert HL1)

  apply (subst uminus-uminus-ident[symmetric, of d])
  apply (subst JavaInt-div-neg-shift)
  apply simp
  prefer 2
  apply (rule quotient-sign-minus1)

  apply (simp-all add: 1 2 3 4 uminus-homom)
  apply (subst zle-zminus)
  apply (rule 4)
  apply (simp add: neg-not-MinInt)
done
qed

```

This lemma formalizes the second property described in (1) for the division operator (JLS §15.17.2).

Although the JLS does not explicitly mention this, we need not exclude the special case  $n = \text{MinInt}$  and  $d = (1::'a)$  here because we already assume that the signs of  $n$  and  $d$  are different.

```

lemma quotient-sign-minus :
  assumes 1: abs (Rep-JavaInt d) ≤ abs (Rep-JavaInt n)
  — we have to regard the values, no wraparound!
  assumes 2: neg (Rep-JavaInt n) ≠ neg (Rep-JavaInt d)
  assumes 3: n ≠ MinInt
  assumes 4: d ≠ 0
  shows (n div d) < 0

```

**proof** –

```

{
  fix x
  assume h1: 0 ≤ (x::int)
  assume h2: neg x
  have False

```

```

    apply (insert h1 h2)
    apply (simp add: neg-eq-less-0)
  done
} note HL1 = this

{
  assume h1: 0 < Rep-JavaInt d
  have Rep-JavaInt n < 0
    apply (insert 2 h1)
    apply (simp add: neg-eq-less-0)
  done
} note HL2 = this

{
  assume h1: Rep-JavaInt d < 0
  have 0 < Rep-JavaInt n
    apply (insert 2 h1)
    apply (simp add: neg-eq-less-0)
    apply (simp add: linorder-not-less)
    apply (insert 1 4)
    apply (simp only: Rep-JavaInt-inject[symmetric])
    apply (simp only: zero-def Abs-JavaInt-inverse zero-in-JavaInt)
    apply (frule abs-neq0-greater-neq0, simp)
    apply (simp add: abs-def)
  done
} note HL3 = this

show ?thesis

apply (insert 1 2)
apply (drule abs-explain-neq, simp)
apply (insert 4)
apply (simp add: Rep-JavaInt-inject[symmetric] zero-def)
apply (simp add: zero-def[symmetric])

apply auto
apply (simp-all add: neg-eq-less-0)
apply (erule swap, simp)
apply (frule HL3)
apply (rule quotient-sign-minus1, simp, simp, simp)

apply (simp add: linorder-not-less)
apply (simp only: HOL.neq-commute[of Rep-JavaInt d 0::int])
apply (drule conjI[of 0 ≤ Rep-JavaInt d 0 ≠ Rep-JavaInt d], simp)
apply (drule order-less-le[symmetric, THEN iffD1])
apply (frule HL2)
apply (rule quotient-sign-minus2, simp, simp, simp)
defer
apply (simp add: linorder-not-less [of Rep-JavaInt n 0])

```

```

apply (drule zle-anti-sym[of Rep-JavaInt n 0], simp)
apply (insert 1, simp)
apply (simp only: zero-less-abs-iff[symmetric, of Rep-JavaInt d])
apply (insert 3)
apply (simp add: not-MinInt-not-MinInt-int)
done
qed

```

## 16.5 Bounds on the Remainder

The following lemmas mirror the corresponding laws from IntDiv which are

$$?b < 0 \implies ?b < ?a \text{ mod } ?b$$

$$?b < 0 \implies ?a \text{ mod } ?b \leq 0$$

$$0 < ?b \implies ?a \text{ mod } ?b < ?b$$

$$0 < ?b \implies 0 \leq ?a \text{ mod } ?b$$

**lemma** *neg-mod-bound*:

**assumes** 1:  $b < 0$

**assumes** 2:  $b \neq \text{MinInt}$

**shows**  $(b::\text{JavaInt}) < a \text{ mod } b$

**proof** –

```

{
assume 1: Rep-JavaInt b < 0
have 0 < - Rep-JavaInt b + Rep-JavaInt a mod Rep-JavaInt b
apply (simp only: zadd-commute[of - Rep-JavaInt b -])
apply (simp only: zdiff-zless-eq[symmetric, of 0 - - Rep-JavaInt b])
apply (simp )
apply (rule IntDiv.neg-mod-bound)
apply (simp add: 1)
done
} note HL1 = this

```

```

have HL2: -1 - MaxInt-int = - (MaxInt-int + 1)
by auto

```

```

{
assume 1: Rep-JavaInt b < 0
have - Rep-JavaInt b + Rep-JavaInt a mod Rep-JavaInt b ≤ MaxInt-int
apply (simp only: zadd-commute[of - Rep-JavaInt b -])
apply (simp only: zdiff-def[symmetric])
apply (simp only: zdiff-zle-eq[of - Rep-JavaInt b MaxInt-int])
apply (insert MinInt-leq [of b])
apply (drule zadd-zle-mono1[of MinInt-int Rep-JavaInt b MaxInt-int])
apply (simp add: JavaInt-maxmin-compare2)
apply (insert 1)
apply (drule IntDiv.neg-mod-sign[of Rep-JavaInt b Rep-JavaInt a])

```

```

apply (cases  $-1 < \text{Rep-JavaInt } b + \text{MaxInt-int}$ )
apply simp

apply (simp add: zle-def [symmetric])
apply (drule order.order-antisym[of -1 Rep-JavaInt b + MaxInt-int], simp)
apply (simp add: zdiff-eq-eq [symmetric, of -1 Rep-JavaInt b MaxInt-int])
apply (simp only: HL2)
apply (simp add: JavaInt-maxmin-compare)
apply (insert 2)
apply (simp add: Rep-JavaInt-inject[symmetric, of b MinInt])
apply (simp add: MinInt-def)
done
} note HL3 = this

{
assume H1: Rep-JavaInt b < 0
have  $-\text{Rep-JavaInt } b + \text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b \in \text{JavaInt}$ 
apply (simp add: JavaInt-def)
apply (rule conjI)
apply (insert MinInt-leq0 H1)
apply (drule HL1)
apply (simp only: order-less-imp-le)
apply (rule HL3, simp)
done
} note HL4 = this

show ?thesis

— this way, we can cleanly distinguish the two cases of the mod definition:
apply (subst mod-def)
apply (cases (((0 < a) & (b < 0)) | ((a < 0) & (0 < b))) &
 $\sim (EX z. ((\text{Rep-JavaInt } a) = (z * (\text{Rep-JavaInt } b))))$ )
apply simp

defer 1
apply (subst if-not-P, simp)

apply (simp-all only: Int-to-JavaInt-ident mod-in-JavaInt)
apply (insert 1 2)
apply (simp-all add: JavaInt-less-def JavaInt-le-def Rep-JavaInt-inject[symmetric]
zero-def MinInt-def)
apply (simp-all add: order-less-le[symmetric])

apply (subst add-def)
apply (simp add: Abs-JavaInt-inverse)
apply (simp add: uminus-def)
apply (simp add: Int-to-JavaInt-ident minus-in-JavaInt)
apply (subst Int-to-JavaInt-ident)

```

```

apply (rule HL4, simp)

apply (subst Abs-JavaInt-inverse)
apply (rule HL4, simp)

apply (insert HL1)
apply simp
done
qed

lemma pos-mod-bound:
  assumes 1:  $0 < b$ 
  shows  $(a::JavaInt) \bmod b < b$ 
proof –

  have L1:  $(0::int) < Rep-JavaInt\ b \implies$ 
     $Rep-JavaInt\ a \bmod Rep-JavaInt\ b - Rep-JavaInt\ b \in JavaInt$ 
  apply (rule mod-minus-in-JavaInt [of b a])
  apply (simp add: MinInt-int-def BitLength-def)
  done

  have HL2:  $-1 - MaxInt-int = - (MaxInt-int + 1)$ 
  by auto

have HL5:  $0 < Rep-JavaInt\ b \implies - Rep-JavaInt\ b \in JavaInt$ 
  apply (rule minus-in-JavaInt)
  apply (insert MinInt-less0)
  apply (drule order-less-trans[of MinInt-int 0 Rep-JavaInt b], simp)
  apply (simp)
  done

  show ?thesis
  apply (insert 1)
  apply (simp add: mod-def le-quasi-def)
  apply (rule conjI)
  apply (rule impI)
  prefer 2
  apply (rule impI)
  apply (thin-tac ?X  $\longrightarrow$  ?Y)
  prefer 2
  apply (thin-tac ?X & ?Y)
  defer 1
  apply (rule IntDiv.pos-mod-bound)
  apply (simp-all add: zero-def)
  apply (subst add-def)
  apply (simp add: Abs-JavaInt-inverse)

```



```

apply (simp add: uminus-def)
apply (simp add: Int-to-JavaInt-ident [of - Rep-JavaInt b] HL5)
apply (subst Int-to-JavaInt-ident)
apply (rule mod-minus-divisor-in-JavaInt, simp)
apply (subst Abs-JavaInt-inverse)
apply (rule mod-minus-divisor-in-JavaInt, simp)

apply auto
apply (subgoal-tac Rep-JavaInt b < 2 * Rep-JavaInt b)
apply (drule IntDiv.pos-mod-bound [of Rep-JavaInt b Rep-JavaInt a])
apply (blast intro: zless-trans)
apply auto
done
qed

```

The phrases stated in (3) in the JLS description of the modulo operator (JLS §15.17.3) are not at all clear to us. We formalized them as follows in `neg-mod-sign` and `pos-mod-sign`, in the hope of meeting the intentions of the authors:

**lemma** *neg-mod-sign*:

```

assumes 1:  $a < 0$ 
assumes 2:  $b \neq 0$ 
shows    ( $a::\text{JavaInt}$ ) mod  $b \leq 0$ 

```

**proof** –

```

{
assume 1:  $a < 0$ 
assume 2:  $b < 0$ 
have    ( $a::\text{JavaInt}$ ) mod  $b \leq 0$ 
apply (insert 1 2)
apply (simp add: mod-def)
apply (auto)
prefer 3
apply (simp-all add: zero-def JavaInt-le-def le-quasi-def)
done
} note neg-mod-sign1 = this

```

```

{
assume 1:  $0 < b$ 
assume 2:  $a < 0$ 
have    ( $a::\text{JavaInt}$ ) mod  $b \leq 0$ 
apply (insert 1 2)
apply (cases EX z. Rep-JavaInt a = z * (Rep-JavaInt b))
apply (subst mod-eq-0-iff2, simp)
apply simp
apply (simp add: mod-def)
apply (simp add: uminus-def zero-def JavaInt-le-def le-quasi-def)
apply (subst Int-to-JavaInt-ident, simp)

```

```

apply (simp only: add-def)
apply (subst Abs-JavaInt-inverse, simp) +
apply (simp only: zadd-commute)
apply (subst Int-to-JavaInt-ident, simp)
apply (subst Abs-JavaInt-inverse, simp)

  apply (simp only: zadd-commute [of - Rep-JavaInt b Rep-JavaInt a mod
Rep-JavaInt b]
    zdiff-def[symmetric])
apply (subst zle-iff-zdiff-zle-0[symmetric])
apply (subst int-le-less)
apply (rule disjI1)
apply (rule IntDiv.pos-mod-bound)
apply simp
done
} note neg-mod-sign2 = this

show ?thesis
apply (insert 1 2)
apply (cases b < 0)
prefer 2
apply (drule linorder-not-less [THEN iffD1])
apply (drule order-le-less [THEN iffD1])
apply (auto simp: neg-mod-sign1 neg-mod-sign2)
done
qed

lemma pos-mod-sign:
assumes 1:  $0 \leq a$ 
assumes 2:  $b \neq 0$ 
assumes 3:  $b \neq \text{MinInt}$ 
shows  $0 \leq (a::\text{JavaInt}) \bmod b$ 
proof -
{
assume 1:  $0 \leq a$ 
assume 2:  $0 < b$ 
have  $0 \leq (a::\text{JavaInt}) \bmod b$ 
apply (insert 1 2)
apply (cases a=0)
apply simp
apply (simp add: mod-def)
apply (auto)
apply (drule order-less-imp-not-less [of 0 b], simp)

apply (drule order-less-imp-not-less [of a 0])
apply (drule order-le-less [of 0 a, THEN iffD1])
apply (simp only: HOL.neq-commute)

```

```

apply simp

apply (simp only: linorder-not-less[of 0 a])
apply (drule order-antisym [of 0 a], simp)
apply (simp add: HOL.neq-commute)

apply (simp only: zero-def JavaInt-le-def le-quasi-def)
apply (subst Abs-JavaInt-inverse[of Rep-JavaInt a mod Rep-JavaInt b])
apply (simp only: mod-in-JavaInt)
apply (simp only: Abs-JavaInt-inverse[of 0] zero-in-JavaInt) +
apply (rule IntDiv.pos-mod-sign)
apply (assumption)
apply (simp add: zero-def)
done
} note pos-mod-sign1 = this

{
assume 1:  $0 \leq a$ 
assume 2:  $b < 0$ 
assume 3:  $b \neq \text{MinInt}$ 
have  $0 \leq (a::\text{JavaInt}) \bmod b$ 
apply (insert 1 2 3)
apply (cases a = 0)
apply simp
apply (drule order-le-imp-less-or-eq)
apply simp
apply (cases EX z. Rep-JavaInt a = z * (Rep-JavaInt b))
apply (subst mod-eq-0-iff2)
apply auto
apply (simp add: mod-def)

apply (simp-all add: uminus-def zero-def le-quasi-def MinInt-def)
apply (simp add: Rep-JavaInt-inject[symmetric])
apply (subst Int-to-JavaInt-ident)
apply (rule minus-in-JavaInt, simp)
apply (simp add: add-def)
apply (subst Abs-JavaInt-inverse)
apply (rule minus-in-JavaInt, simp)
apply (subst Int-to-JavaInt-ident)
apply simp
apply (rule mod-minus-in-JavaInt, simp)

apply (simp add: JavaInt-le-def)
apply (subst Abs-JavaInt-inverse)
apply (rule mod-minus-in-JavaInt, simp)
apply (simp add: zle-zdiff-eq)
apply (subst int-le-less)
apply (rule disjI1)
apply (rule IntDiv.neg-mod-bound)

```

```

    apply assumption
  done
} note pos-mod-sign2 = this

show ?thesis
  apply (insert 1 2 3)
  apply (cases b < 0)
  prefer 2
  apply (drule linorder-not-less [THEN iffD1])
  apply (tactic rotate-tac ~1 1)
  apply (drule order-le-less [THEN iffD1])
  apply (auto simp: pos-mod-sign1 pos-mod-sign2)
  done
qed

```

This lemma formalizes the property described in (4) for the modulo operator (JLS §15.17.3).

It is not clear whether the “magnitude of the result” refers to the mathematical absolute value or to the Java method `Math.abs`. We decided to use the function `abs` on `JavaInt` which allows us to stay in the abstract model. This has the drawback that the lemma cannot be used for `b = MinInt` because  $|MinInt| = MinInt$ , therefore there exists no `JavaInt` value which is strictly smaller than  $|MinInt|$ .

```

lemma JavaInt-mod-less:
  assumes 1: b ≠ 0
  assumes 2: b ≠ MinInt
  shows   abs ((a::JavaInt) mod b) < abs b
proof -

```

```

{
  assume L5-1: (0 ≤ a)
  assume L5-2: (b < 0)
  have a mod b < (- b)
    apply (insert 1 2 L5-1 L5-2)
    apply (subst JavaInt-mod-neg2[symmetric])
    apply (rule pos-mod-bound)
    apply (rule uminus-pos-if-neg, simp+)
  done
} note L5 = this

{
  assume L6-2: (0 < b)
  have a mod b < b
    apply (insert L6-2)
    apply (rule pos-mod-bound, simp)
  done
} note L6 = this

```

```

{
fix a b
assume L3-1: (a < (0::JavaInt))
assume L3-2: (b < (0::JavaInt))
assume L3-3: (b ≠ 0)
assume L3-4: (b ≠ MinInt)

have - (a mod b) < (- b)
  apply (insert L3-1 L3-2 L3-3 L3-4)
  apply (insert MinInt-is-least2[of b])
  apply (tactic rotate-tac ~1 1)
  apply (drule order-le-less [THEN iffD1])
  apply auto
  apply (subst swap-le)
  apply (insert MinInt-is-least2[of b])
  apply (simp add: less-vs-eq-or-le less-vs-not-le[symmetric])
  apply (auto)
  prefer 2
  apply (rule neg-mod-bound)
  apply auto
  apply (drule neg-mod-bound [of b a], simp)
  apply auto
  done
} note L3 = this

{
assume L4-1: (a < 0)
assume L4-2: (0 < b)
have - (a mod b) < b
  apply (subst JavaInt-mod-neg2[symmetric])
  apply (subst uminus-uminus-ident[symmetric, of b])
  apply (subst swap-le[of a mod (- (- (- b))) - b])
  apply (simp-all)
  — discharge 1st condition:
  apply (insert L4-2 2)
  apply (drule uminus-neg-if-pos[of b])
  apply (simp add: neg-not-MinInt[symmetric, of b])
  apply (drule neg-mod-bound [of - b a], simp)
  apply auto
  — discharge 2nd condition:
  apply (simp add: neg-not-MinInt[symmetric, of b])
  — discharge rest:
  apply (drule uminus-neg-if-pos[of b])
  apply (simp add: neg-not-MinInt[symmetric, of b])
  apply (drule neg-mod-bound [of - b a], simp+)
  done
} note L4 = this

```

```

{
assume L1-1:  $0 \leq a$ 
have abs (a mod b) < abs b
  apply (insert 1 2 L1-1)
  apply (subst abs-pos [of a mod b])
  apply (auto intro: pos-mod-sign)
  apply (cases  $0 \leq b$ )
  apply (auto simp: abs-neg linorder-not-le)
  apply (subst abs-pos [of b], simp)
  apply (simp add: order-le-less [of 0 b])
  prefer 2
  apply (subst abs-neg)
  apply (simp add: order-less-imp-le)
  apply (drule L5, simp+)
  apply (drule L6, simp+)
done
} note L1 = this

```

```

{
assume L2-1:  $\sim (0 \leq a)$ 
have abs (a mod b) < abs b
  apply (insert 1 2 L2-1)
  apply (drule linorder-not-le [THEN iffD1])
  — now resolve abs'es
  apply (subst abs-neg [of a mod b])
  apply (auto intro: neg-mod-sign)
  apply (cases  $0 \leq b$ )
  apply (auto simp: abs-pos linorder-not-le)
  apply (simp add: order-le-less)
  prefer 2
  apply (subst abs-neg)
  apply (simp add: order-less-imp-le)
  apply (drule L3, simp+)
  apply (drule L4, simp+)
done
} note L2 = this
show ?thesis
  apply (insert 1 2)
  apply (cases  $0 \leq a$ )
  apply (drule L1, simp)
  apply (drule L2, simp)
done
qed

```

## 16.6 Main Theorem: The div/mod lemma

lemma *MinInt-minusone-div-mod-eq* :

$$(\text{MinInt div } -1) * (-1) + (\text{MinInt mod } -1) = \text{MinInt}$$

by (*simp add: MinInt-mod-minusone MinInt-div-minusone  
MinInt-times-minusone*)

**lemma** *eq-znull-eq-jnull:*

$(\text{Rep-JavaInt } a = (0::\text{int})) = (a = (0::\text{JavaInt}))$

by (*simp add: Rep-JavaInt-inject [symmetric] Abs-JavaInt-inverse zero-def*)

This lemma formalizes the property described in (1) for the modulo operator (JLS §15.17.3).

**lemma** *JavaInt-div-mod-result :*

$((a::\text{JavaInt}) \text{ div } b) * b + (a \text{ mod } b) = a$

**proof** –

**have** \* :  $(((((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ div } (\text{Rep-JavaInt } b))) * b) + b) + ((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ mod } (\text{Rep-JavaInt } b))) - b)) = (((((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ div } (\text{Rep-JavaInt } b))) * b) + ((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ mod } (\text{Rep-JavaInt } b))))))$

**proof** –

**have**  $(((((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ div } (\text{Rep-JavaInt } b))) * b) + b) + ((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ mod } (\text{Rep-JavaInt } b))) + - b)) = (((((\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ div } (\text{Rep-JavaInt } b))) * b) + b) + ((- b) + (\text{Int-to-JavaInt } ((\text{Rep-JavaInt } a) \text{ mod } (\text{Rep-JavaInt } b))))))$   
*(is - = (?x + b) + ((- b) + ?y))*

by (*simp only: JavaInt-add-commute*)

**also have** ... =  $?x + (b + ((- b) + ?y))$

by (*simp add: JavaInt-add-assoc*)

**also have** ... =  $?x + (b + (- b) + ?y)$

by (*simp add: JavaInt-add-assoc [symmetric]*)

**also have** ... =  $?x + ?y$

by *simp*

**also from** *calculation show ?thesis by auto*

**qed**

**moreover**

{ **fix** *a b*

**have**

$\text{Rep-JavaInt } (\text{Int-to-JavaInt } (\text{Rep-JavaInt } (\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b * \text{Rep-JavaInt } b) + \text{Rep-JavaInt } (\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)))) = \text{Rep-JavaInt } a$

**proof** –

**have**

$\text{Rep-JavaInt } (\text{Int-to-JavaInt } (\text{Rep-JavaInt } (\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b * \text{Rep-JavaInt } b) +$

```

Rep-JavaInt (Int-to-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)))) =
((Rep-JavaInt a div Rep-JavaInt b * Rep-JavaInt b - MinInt-int)
 mod - ((2::int) * MinInt-int) + MinInt-int +
((Rep-JavaInt a mod Rep-JavaInt b - MinInt-int)
 mod - ((2::int) * MinInt-int) + MinInt-int) -
MinInt-int) mod - ((2::int) * MinInt-int) + MinInt-int
(is Rep-JavaInt (Int-to-JavaInt (Rep-JavaInt (Int-to-JavaInt ?rdiv)
 + Rep-JavaInt (Int-to-JavaInt ?rmod)))) =
( (?rdiv - ?mi) mod ?zmi + ?mi +
((?rmod - ?mi) mod ?zmi + ?mi)
 - ?mi) mod ?zmi + ?mi)
apply (simp add: Int-to-JavaInt-def)
apply (subst Rep-Abs-remove)+
apply auto
done
moreover have
... = ( (?rdiv - ?mi) mod ?zmi +
((?rmod - ?mi) mod ?zmi + ?mi + ?mi)
 - ?mi) mod ?zmi + ?mi
by (simp add: zadd-commute)
moreover have
... = ( (?rdiv - ?mi) mod ?zmi +
(?rmod - ?mi) mod ?zmi + ?mi + ?mi
 - ?mi) mod ?zmi + ?mi
by (simp add: zadd-assoc)
moreover have
... = ( (?rdiv - ?mi) mod ?zmi +
(?rmod - ?mi) mod ?zmi + ?mi
) mod ?zmi + ?mi
by (simp add: zadd-commute)
moreover have
... = ( ((?rdiv - ?mi) mod ?zmi +
(?rmod - ?mi) mod ?zmi)
 + ?mi
) mod ?zmi + ?mi
(is - = ( ?rdivmod + ?mi ) mod ?zmi + ?mi)
by auto
moreover have
... = ( (?rdivmod mod ?zmi)
 + ?mi
) mod ?zmi + ?mi
by (subst zmod-zadd-left-eq[symmetric], auto)
moreover have ... = ( ((?rdiv - ?mi + ?rmod - ?mi) mod ?zmi)
 + ?mi
) mod ?zmi + ?mi
by (simp add: zmod-zadd1-eq [symmetric])
moreover have ... = ( ((Rep-JavaInt b * (Rep-JavaInt a div Rep-JavaInt b)
 + ?rmod - ?mi - ?mi) mod ?zmi)

```



```

+ ?mi
) mod ?zmi + ?mi
by (simp add: zadd-commute zmult-commute)
moreover have ... = ( ((Rep-JavaInt a - ?mi - ?mi) mod ?zmi)
+ ?mi
) mod ?zmi + ?mi
by (simp add: zmod-zdiv-equality [symmetric])
moreover have ... = ( Rep-JavaInt a - ?mi - ?mi
+ ?mi
) mod ?zmi + ?mi
by (simp del: Minustwo-Minus-two add: zmod-zadd-left-eq [symmetric])
moreover have ... = ( Rep-JavaInt a - ?mi ) mod ?zmi + ?mi
by (simp add: zadd-commute zadd-zminus-inverse2)
moreover have ... = Rep-JavaInt a
apply (subst Rep-Abs-remove [symmetric])
apply (simp-all only: Rep-JavaInt-inject
zdiff-def zmult-zminus-right [symmetric]
Int-to-JavaInt-def [THEN meta-eq-to-obj-eq, symmetric]
Int-to-JavaInt-ident Rep-JavaInt Rep-JavaInt-inverse)
done
ultimately show ?thesis by auto
qed
} note proof-core = this

```

**moreover** — proof for special case

```

{ fix a b
have Rep-JavaInt b = 0  $\implies$ 
Rep-JavaInt
(Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
Rep-JavaInt a
apply (simp add: IntDiv.DIVISION-BY-ZERO)
apply (simp add: Rep-JavaInt)
done
} note zero-proof = this

```

**moreover** — main proof of one of the four subgoals

```

{ fix a b
have
[[ 0 ≤ Rep-JavaInt a; Rep-JavaInt b < 0 ]]  $\implies$ 
Rep-JavaInt
(Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
Rep-JavaInt a
proof —
assume 1: 0 ≤ Rep-JavaInt a
assume 2: Rep-JavaInt b < 0

have fold1: Rep-JavaInt

```

```

(Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
Int-to-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
Rep-JavaInt
(Int-to-JavaInt
(Rep-JavaInt (Int-to-JavaInt
(Rep-JavaInt a div Rep-JavaInt b * Rep-JavaInt b)) +
Rep-JavaInt (Int-to-JavaInt
(Rep-JavaInt a mod Rep-JavaInt b))))
apply (insert 1 2)
apply (simp add: times-def add-def)
apply (simp add: Int-to-JavaInt-ident
div-Int-Abs-ident [of a b]
MinInt-int-def BitLength-def)
apply (simp add: Abs-JavaInt-inverse
div-in-JavaInt [of a b]
MinInt-int-def BitLength-def)
done

then show ?thesis
by (simp only: fold1 mod-Int-Abs-ident [symmetric] proof-core)
qed
} note pos-neg-proof = this

```

**moreover** — main proof of one of the four subgoals

```

{ fix a b
have
[[ 0 ≤ Rep-JavaInt a; 0 < Rep-JavaInt b ]] ==>
Rep-JavaInt
(Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
Rep-JavaInt a
proof –
assume 1: 0 ≤ Rep-JavaInt a
assume 2: 0 < Rep-JavaInt b

have fold1: Rep-JavaInt
(Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
Int-to-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
Rep-JavaInt
(Int-to-JavaInt
(Rep-JavaInt (Int-to-JavaInt
(Rep-JavaInt a div Rep-JavaInt b * Rep-JavaInt b)) +
Rep-JavaInt (Int-to-JavaInt
(Rep-JavaInt a mod Rep-JavaInt b))))
apply (insert 1 2)
apply (simp add: times-def add-def)
apply (simp add: Int-to-JavaInt-ident
div-Int-Abs-ident [of a b]
MinInt-int-def BitLength-def)

```

```

apply (simp add: Abs-JavaInt-inverse
        div-in-JavaInt [of a b]
        MinInt-int-def BitLength-def)
done

then show ?thesis
apply (insert 1 2)
apply (simp only: fold1 mod-Int-Abs-ident [symmetric] proof-core)
done
qed
} note nonneg-pos-proof = this

moreover — main proof of one of the four subgoals
{ fix a b
  have
    [[ Rep-JavaInt a < 0; 0 < Rep-JavaInt b ]] ==>
    Rep-JavaInt
    (Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
     Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
    Rep-JavaInt a
  proof –
    assume 1: Rep-JavaInt a < 0
    assume 2: 0 < Rep-JavaInt b

    have fold1: Rep-JavaInt
      (Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
       Int-to-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
      Rep-JavaInt
      (Int-to-JavaInt
       (Rep-JavaInt (Int-to-JavaInt
        (Rep-JavaInt a div Rep-JavaInt b * Rep-JavaInt b)) +
        Rep-JavaInt (Int-to-JavaInt
         (Rep-JavaInt a mod Rep-JavaInt b))))
    apply (insert 1 2)
    apply (simp add: times-def add-def)
    apply (simp add: Int-to-JavaInt-ident
            div-Int-Abs-ident [of a b]
            MinInt-int-def BitLength-def)
    apply (simp add: Abs-JavaInt-inverse
            div-in-JavaInt [of a b]
            MinInt-int-def BitLength-def)
    done

    then show ?thesis
      by (simp only: fold1 mod-Int-Abs-ident [symmetric] proof-core)
    qed
} note neg-pos-proof = this

moreover — groups nonneg-pos-proof and neg-pos-proof

```

```

{ fix a b
  have
     $\llbracket 0 < \text{Rep-JavaInt } b \rrbracket \implies$ 
     $\text{Rep-JavaInt}$ 
     $(\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) * b +$ 
     $\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)) =$ 
     $\text{Rep-JavaInt } a$ 
  proof -
    assume 1:  $0 < \text{Rep-JavaInt } b$ 
    show ?thesis
      apply (insert 1)
      apply (cases  $0 \leq \text{Rep-JavaInt } a$ )
      apply (simp only: mod-Int-Abs-ident)
      apply (rule nonneg-pos-proof, auto)
      apply (simp only: JavaInt-times-commute [of b -])
      apply (subst JavaInt-add-commute)
      apply (rule neg-pos-proof)
      apply auto
    done
  qed
} note all-pos-proof = this

moreover — main proof of one of the four subgoals
{ fix a b
  have
     $\llbracket \text{Rep-JavaInt } a < 0; \text{Rep-JavaInt } b < 0; \text{Rep-JavaInt } a \neq \text{MinInt-int} \mid$ 
 $\text{Rep-JavaInt } b \neq -1 \rrbracket$ 
     $\implies$ 
     $\text{Rep-JavaInt}$ 
     $(\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) * b +$ 
     $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)) =$ 
     $\text{Rep-JavaInt } a$ 
  proof -
    assume 1:  $\text{Rep-JavaInt } a < 0$ 
    assume 2:  $\text{Rep-JavaInt } b < 0$ 
    assume 3:  $\text{Rep-JavaInt } a \neq \text{MinInt-int} \mid \text{Rep-JavaInt } b \neq -1$ 

    have fold1:  $\text{Rep-JavaInt}$ 
       $(\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) * b +$ 
       $\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)) =$ 
       $\text{Rep-JavaInt}$ 
       $(\text{Int-to-JavaInt}$ 
       $(\text{Rep-JavaInt } (\text{Int-to-JavaInt}$ 
       $(\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b * \text{Rep-JavaInt } b)) +$ 
       $\text{Rep-JavaInt } (\text{Int-to-JavaInt}$ 
       $(\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b))))$ 
      apply (insert 1 2 3)
      apply (simp add: times-def add-def)
      apply (simp add: Int-to-JavaInt-ident

```

```

    div-Int-Abs-ident [of a b]
    MinInt-int-def BitLength-def)
  apply (simp add: Abs-JavaInt-inverse
    div-in-JavaInt [of a b]
    MinInt-int-def BitLength-def)
done

then show ?thesis
  by (simp only: fold1 mod-Int-Abs-ident [symmetric] proof-core)
qed
} note neg-neg-proof = this

moreover — proof of special case MinInt div -1
have MinInt-minusone-proof:
  [[ Rep-JavaInt a = MinInt-int; Rep-JavaInt b = -1 ]] ==>
  Rep-JavaInt
  (Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
  Int-to-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
  Rep-JavaInt a
proof —
  assume 1: Rep-JavaInt a = MinInt-int
  assume 2: Rep-JavaInt b = -1
  show ?thesis
    apply (simp add: 1 2 times-def Int-to-JavaInt-def add-def)
    apply (simp add: Abs-JavaInt-inverse
      JavaInt-def MaxInt-int-def MinInt-int-def
      BitLength-def) +
  done
qed

moreover
{ fix a b
  have
    [[ Rep-JavaInt a ≤ 0 ]] ==>
    Rep-JavaInt
    (Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
    Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
    Rep-JavaInt a
  proof —
    assume 1: Rep-JavaInt a ≤ 0
    show ?thesis
      apply (cases Rep-JavaInt a < 0)
      apply (cases Rep-JavaInt b < 0)
      apply (cases Rep-JavaInt a ≠ MinInt-int)
      apply (rule neg-neg-proof, auto)
      apply (cases Rep-JavaInt b = -1)
      — Ra = MinInt, Rb = (-1::'a)
      apply (simp add: MinInt-minusone-proof)
  }

```

```

apply (simp add: times-def)
apply (simp add: Int-to-JavaInt-def)
apply (simp add: MinInt-def)
apply (simp add: Minustwo-Minus-two zero-def[symmetric]) +
—  $Ra = \text{MinInt}, Rb \neq (-1::'a)$ 
apply (tactic res-inst-tac [(t,MinInt-int)] subst 1)
apply (tactic atac 1)
apply (subst JavaInt-times-commute)
apply (subst JavaInt-add-commute)
apply (rule neg-neg-proof)
apply simp-all

—  $Ra < (0::'a); (0::'a) \leq Rb$ 
apply (cases Rep-JavaInt b = 0)
apply (subst JavaInt-times-commute [of b -])
apply (subst JavaInt-add-commute)
apply (rule zero-proof, auto)

apply (subst JavaInt-times-commute [of b -])
apply (subst JavaInt-add-commute)
apply (rule neg-pos-proof, auto)
apply (insert 1)
apply (simp add: zle-def [symmetric])
apply (drule zle-anti-sym, simp)
apply auto
done
qed
} note nonpos-all-proof = this

moreover
{ fix a b
  have
     $\llbracket 0 \leq \text{Rep-JavaInt } a \rrbracket \implies$ 
     $\text{Rep-JavaInt}$ 
     $(\text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) * b +$ 
     $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b)) =$ 
     $\text{Rep-JavaInt } a$ 
  proof —
    assume 1:  $0 \leq \text{Rep-JavaInt } a$ 
    show ?thesis
      apply (insert 1)
      apply (cases  $0 < \text{Rep-JavaInt } b$ )
      apply (rule nonneg-pos-proof, auto)
      apply (cases  $\text{Rep-JavaInt } b = 0$ )
      apply (subst JavaInt-times-commute [of b -])
      apply (subst JavaInt-add-commute)
      apply (rule zero-proof, auto)
      apply (rotate-tac)
      apply (simp add: zle-def [symmetric])

```

```

    apply (drule zle-imp-zless-or-eq, simp)
    apply (subst JavaInt-times-commute [of b -])
    apply (subst JavaInt-add-commute)

    apply (rule pos-neg-proof, auto)

  done
qed
} note nonneg-all-proof = this

moreover
{ fix a b
  have Rep-JavaInt
    (Int-to-JavaInt (Rep-JavaInt a div Rep-JavaInt b) * b +
     Abs-JavaInt (Rep-JavaInt a mod Rep-JavaInt b)) =
    Rep-JavaInt a
    apply (cases 0 ≤ Rep-JavaInt a)
    apply (rule nonneg-all-proof)
    apply (simp add: zle-def)
    apply (rule nonpos-all-proof, auto)
  done
} note all-all-proof = this

moreover
{ fix a b
  have ∧ z. [ Rep-JavaInt a = z * Rep-JavaInt b;
    Rep-JavaInt b ≠ 0;
    Rep-JavaInt a ≠ MinInt-int | Rep-JavaInt b ≠ -1 ]
    ⇒ Rep-JavaInt (Int-to-JavaInt z * b) = z * Rep-JavaInt b
    apply (simp add: times-def)
    apply (subgoal-tac z ∈ JavaInt)
    apply (simp add: Int-to-JavaInt-ident)
    apply (drule HOL.eq-commute[THEN iffD1])
    apply (rotate-tac -1)
    apply (erule ssubst)
    apply (simp add: Int-to-JavaInt-ident Rep-JavaInt)
    apply (simp add: Factor-in-JavaInt)
  done
} note LZ1 = this

moreover
{ fix a b
  assume 1: Rep-JavaInt a = z * Rep-JavaInt b
  assume 2: Rep-JavaInt b ≠ 0
  assume 3: Rep-JavaInt a ≠ MinInt-int | Rep-JavaInt b ≠ -1
  have Rep-JavaInt (Int-to-JavaInt z * b) = z * Rep-JavaInt b
    apply (simp add: times-def)
    apply (subgoal-tac z ∈ JavaInt)
    apply (simp add: Int-to-JavaInt-ident)

```

```

apply (insert 1)
apply (drule HOL.eq-commute[THEN iffD1])
apply (rotate-tac -1)
apply (erule ssubst)
apply (simp add: Int-to-JavaInt-ident Rep-JavaInt)
apply (insert 2 3)
apply (simp add: Factor-in-JavaInt)
done
} note LZ1-noquant = this

```

**moreover**

```

have LZ2:  $\bigwedge z::int.$ 
   $\llbracket \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b; \text{Rep-JavaInt } b \neq (0::int) \rrbracket$ 
   $\implies \text{Rep-JavaInt } (\text{Int-to-JavaInt } z * b) = z * \text{Rep-JavaInt } b$ 
apply (case-tac Rep-JavaInt a = MinInt-int)
apply (case-tac Rep-JavaInt b = -1)
— solve problem for special case MinInt-int and  $-1::'a$ 
apply simp
apply (rotate-tac 1)
apply (simp add: HOL.eq-commute)
apply (simp add: equation-zminus)

apply (simp add: times-def)
apply (simp add: Int-to-JavaInt-def MinInt-def Minustwo-Minus-two)
— proved special case

apply (rule LZ1, auto) +
done

```

**moreover**

```

have SG1:  $0 < a \ \& \ b < 0$ 
 $\implies ((\forall z. \text{Rep-JavaInt } a \neq z * \text{Rep-JavaInt } b) \longrightarrow$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a) \ \&$ 
   $((\exists z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b) \longrightarrow$ 
   $((\forall z. \text{Rep-JavaInt } a \neq z * \text{Rep-JavaInt } b) \longrightarrow$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a) \ \&$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \bmod \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a)$ 
apply auto
apply (subst Rep-JavaInt-inject [symmetric])
apply (subst JavaInt-times-commute [of b -])
apply (subst JavaInt-add-commute)
apply (rule all-all-proof)
apply (simp add: Rep-JavaInt-inject [symmetric])

```



```

      zero-def Abs-JavaInt-inverse)
apply (simp add: zero-def[symmetric])
apply (simp add: Rep-JavaInt-inject [symmetric])
apply (simp add: LZ2)
done

moreover
have SG2b:  $\bigwedge z::int.$ 
   $\llbracket \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b; \text{Rep-JavaInt } b \neq (0::int) \rrbracket$ 
   $\implies \text{Int-to-JavaInt } z * b = a$ 
apply (simp add: Rep-JavaInt-inject [symmetric])
apply (rule LZ2, auto)
done

moreover
have SG2:  $0 < a \longrightarrow \sim b < 0$ 
   $\implies ((0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b) \ \& \ (\forall z. \text{Rep-JavaInt } a \neq z * \text{Rep-JavaInt } b)) \longrightarrow$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a \ \&$ 
   $((a < 0 \longrightarrow \sim 0 < b) \longrightarrow$ 
   $b + (\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b)) =$ 
   $a) \ \&$ 
   $((a < 0 \longrightarrow 0 < b \longrightarrow (\exists z. \text{Rep-JavaInt } a = z * \text{Rep-JavaInt } b)) \longrightarrow$ 
   $((0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b) \ \&$ 
   $(\forall z. \text{Rep-JavaInt } a \neq z * \text{Rep-JavaInt } b) \longrightarrow$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a) \ \&$ 
   $\text{Abs-JavaInt } (\text{Rep-JavaInt } a \text{ mod } \text{Rep-JavaInt } b) +$ 
   $b * \text{Int-to-JavaInt } (\text{Rep-JavaInt } a \text{ div } \text{Rep-JavaInt } b) =$ 
   $a)$ 
apply (auto simp: eq-znull-eq-jnull)
prefer 4
apply (simp add: zero-def eq-znull-eq-jnull)
prefer 8
apply (simp add: zero-def eq-znull-eq-jnull)

prefer 4
apply (simp-all add: zero-def[symmetric])
apply (rule SG2b, auto simp: eq-znull-eq-jnull[of b])
prefer 7
apply (rule SG2b, auto simp: eq-znull-eq-jnull[of b])

— now we have 6 equally shaped subgoals
apply (simp-all only: JavaInt-times-commute [of b -])
apply (subst JavaInt-add-commute, subst Rep-JavaInt-inject [symmetric], rule

```

*all-all-proof*)+  
**done**

**moreover** — this is the main theorem – impossible to use ultimately ... directly

**have**  $((a::JavaInt) \text{ div } b) * b + (a \text{ mod } b) = a$   
**apply** (*simp only: div-def mod-def*)  
**apply** (*cases 0 < a & b < 0*)  
**apply** (*simp-all split: split-if*)  
 — 2 subgoals  
**apply** (*rule SG1*)  
**prefer** 2  
**apply** (*rule SG2*)  
**apply** *auto*  
**done**

**ultimately show** *?thesis* **by** *auto*  
**qed**

## 16.7 Calculating Example Values for div and mod

The following lemmas formalize the properties described in (6) for the modulo operator (JLS §15.17.3), i.e. they prove that the example values given in the JLS are indeed met by our formalization.

**lemma** *div-mod-example1* :

$(5::JavaInt) \text{ mod } 3 = 2$   
**apply** (*simp add: mod-def number-of-def*)  
**apply** (*simp add: JavaInt-less-def JavaInt-le-def zero-def diff-def*)  
**apply** (*subgoal-tac 5 ∈ JavaInt*)  
**apply** (*subgoal-tac 3 ∈ JavaInt*)  
**apply** (*subgoal-tac 2 ∈ JavaInt*)  
**apply** (*simp add: Int-to-JavaInt-ident*)  
**apply** (*thin-tac [1–2] ?x ∈ JavaInt*)  
**apply** (*thin-tac ?x ∈ JavaInt*)  
**apply** (*simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def*)  
**done**

**lemma** *div-mod-example2* :

$(5::JavaInt) \text{ div } 3 = 1$   
**apply** (*simp add: div-def number-of-def*)  
**apply** (*simp add: JavaInt-less-def JavaInt-le-def zero-def add-def one-def*)  
**apply** (*subgoal-tac 5 ∈ JavaInt*)  
**apply** (*subgoal-tac 3 ∈ JavaInt*)  
**apply** (*simp add: Int-to-JavaInt-ident*)  
**apply** (*thin-tac ?x ∈ JavaInt*)  
**apply** (*simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def*)  
**done**

```

lemma div-mod-example3 :
  (5::JavaInt) mod -3 = 2
  apply (simp add: mod-def number-of-def splits del: split-if)
  apply (simp add: JavaInt-less-def JavaInt-le-def add-def uminus-def
    zero-def splits del: split-if)
  apply (subgoal-tac 5 ∈ JavaInt)
  apply (subgoal-tac -3 ∈ JavaInt)
  apply (subgoal-tac 2 ∈ JavaInt)
  apply (subgoal-tac 3 ∈ JavaInt)
  apply (simp add: Int-to-JavaInt-ident)
  apply (simp add: Abs-JavaInt-inject)
  apply (simp add: number-of-def)
  apply (insert minusone-in-JavaInt)
  apply (simp only: Int-to-JavaInt-ident)
  apply (simp only: Abs-JavaInt-inject)
  apply simp

  apply (thin-tac [1-5] ?x ∈ JavaInt)
  apply (thin-tac [1-4] ?x ∈ JavaInt)
  apply (thin-tac [1-3] ?x ∈ JavaInt)
  apply (thin-tac [1-2] ?x ∈ JavaInt)
  apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)

  apply (rule HOL.ccontr)
  apply simp
  apply (simp only: HOL.eq-commute [of - 5] zmult-commute [of - -3])
  apply (simp only: zmod-eq-0-iff[symmetric])
  apply auto
done

lemma div-mod-example4 :
  (5::JavaInt) div -3 = -1
  apply (simp add: div-def number-of-def)
  apply (simp add: JavaInt-less-def JavaInt-le-def zero-def one-def add-def)
  apply (subgoal-tac 5 ∈ JavaInt)
  apply (subgoal-tac -3 ∈ JavaInt)
  apply (subgoal-tac -2 ∈ JavaInt)
  apply (simp add: Int-to-JavaInt-ident)
  apply (simp add: Abs-JavaInt-inject)
  apply (simp add: number-of-def)
  apply (insert minusone-in-JavaInt)
  apply (simp only: Int-to-JavaInt-ident)
  apply (simp only: Abs-JavaInt-inject)
  apply (thin-tac [1-4] ?x ∈ JavaInt)
  apply (thin-tac [1-3] ?x ∈ JavaInt)
  apply (thin-tac [1-2] ?x ∈ JavaInt)
  apply (thin-tac ?x ∈ JavaInt)
  apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
  apply (rule HOL.ccontr)

```

```

apply simp
apply (simp only: HOL.eq-commute [of - 5] zmult-commute [of - -3])
apply (simp only: zmod-eq-0-iff [symmetric])
apply auto
done

```

```

lemma div-mod-example5 :
   $(-5::JavaInt) \bmod 3 = -2$ 
apply (simp add: mod-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def add-def uminus-def)
apply (subgoal-tac  $-5 \in JavaInt$ )
apply (subgoal-tac  $3 \in JavaInt$ )
apply (subgoal-tac  $-3 \in JavaInt$ )
apply (subgoal-tac  $-2 \in JavaInt$ )
apply (simp add: Int-to-JavaInt-ident)
apply (simp add: Abs-JavaInt-inject)
apply (thin-tac [1-4]  $?x \in JavaInt$ )
apply (thin-tac [1-3]  $?x \in JavaInt$ )
apply (thin-tac [1-2]  $?x \in JavaInt$ )
apply (thin-tac  $?x \in JavaInt$ )
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
apply (rule HOL.ccontr)
apply simp
apply (simp only: HOL.eq-commute [of - -5] zmult-commute [of - 3])
apply (simp only: zmod-eq-0-iff [symmetric])
apply auto
done

```

```

lemma div-mod-example6 :
   $(-5::JavaInt) \text{div } 3 = -1$ 
apply (simp add: div-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def add-def one-def)
apply (subgoal-tac  $-5 \in JavaInt$ )
apply (subgoal-tac  $3 \in JavaInt$ )
apply (subgoal-tac  $-2 \in JavaInt$ )
apply (simp add: Int-to-JavaInt-ident)
apply (simp add: Abs-JavaInt-inject)
apply (simp add: number-of-def)
apply (insert minusone-in-JavaInt)
apply (simp only: Int-to-JavaInt-ident)
apply (simp only: Abs-JavaInt-inject)
apply (thin-tac [!]  $?x \in JavaInt$ )
apply (thin-tac [1-3]  $?x \in JavaInt$ )
apply (thin-tac [1-2]  $?x \in JavaInt$ )
apply (thin-tac  $?x \in JavaInt$ )
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
apply (rule HOL.ccontr)
apply simp

```

```

apply (simp only: HOL.eq-commute [of - -5] zmult-commute [of - 3])
apply (simp only: zmod-eq-0-iff [symmetric])
apply auto
done

```

```

lemma div-mod-example7 :
   $(-5::JavaInt) \bmod -3 = -2$ 
apply (simp add: mod-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def add-def uminus-def)
apply (subgoal-tac -5 ∈ JavaInt)
apply (subgoal-tac -3 ∈ JavaInt)
apply (subgoal-tac -2 ∈ JavaInt)
apply (simp add: Int-to-JavaInt-ident)
apply (thin-tac [1-2] ?x ∈ JavaInt)
apply (thin-tac ?x ∈ JavaInt)
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
done

```

```

lemma div-mod-example8 :
   $(-5::JavaInt) \operatorname{div} -3 = 1$ 
apply (simp add: div-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def add-def one-def)
apply (subgoal-tac -5 ∈ JavaInt)
apply (subgoal-tac -3 ∈ JavaInt)
apply (simp add: Int-to-JavaInt-ident)
apply (thin-tac ?x ∈ JavaInt)
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
done

```

The following test lemmas were added by us because we found it important to verify the behavior of div and mod in these cases.

```

lemma div-mod-example9 :
   $(8::JavaInt) \operatorname{div} -2 = -4$ 
apply (simp add: div-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def one-def add-def)
apply (subgoal-tac 8 ∈ JavaInt)
apply (subgoal-tac -2 ∈ JavaInt)
apply (subgoal-tac -4 ∈ JavaInt)
apply (subgoal-tac -3 ∈ JavaInt)
apply (simp add: Int-to-JavaInt-ident)
apply (simp add: Abs-JavaInt-inject)
apply (thin-tac [1-3] ?x ∈ JavaInt)
apply (thin-tac [1-2] ?x ∈ JavaInt)
apply (thin-tac ?x ∈ JavaInt)
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
done

```

```

lemma div-mod-example10 :
   $(0::JavaInt) \operatorname{div} -2 = 0$ 

```

```

apply (simp add: div-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def one-def add-def)
done

lemma div-mod-example11 :
  (-8::JavaInt) mod 12 = -8
apply (simp add: mod-def number-of-def)
apply (simp add: JavaInt-less-def JavaInt-le-def zero-def add-def uminus-def)
apply (subgoal-tac -8 ∈ JavaInt)
apply (subgoal-tac 12 ∈ JavaInt)
apply (subgoal-tac -12 ∈ JavaInt)
apply (subgoal-tac 4 ∈ JavaInt)
apply (simp add: Int-to-JavaInt-ident)
apply (simp add: Abs-JavaInt-inject)
apply (thin-tac [1-4] ?x ∈ JavaInt)
apply (thin-tac [1-3] ?x ∈ JavaInt)
apply (thin-tac [1-2] ?x ∈ JavaInt)
apply (thin-tac ?x ∈ JavaInt)
apply (simp-all add: JavaInt-def MinInt-int-def MaxInt-int-def BitLength-def)
apply (rule HOL.ccontr)
apply simp
apply (simp only: HOL.eq-commute [of - -2] zmult-commute [of - 3])
apply (simp only: zmod-eq-0-iff[symmetric])
apply auto
done

```

end

## 17 Formalization With Bitstring Representation

**theory** JavaIntegersBit = JavaIntegersDef :

**lemma** numeral0-is-number-of-pls:  
 (number-of::bin⇒int) bin.Pls = Numeral0  
**by** (auto)

**constdefs**

*bin-of* :: JavaInt ⇒ bin  
*bin-of* x ≡ THE y. x = number-of y

**consts**

$$pad :: [nat, bin] \Rightarrow bin$$
**primrec**

$$\begin{aligned}
pad-0 &: pad\ 0\ x = x \\
pad-Suc &: pad\ (Suc\ k)\ x = \\
&\quad (case\ x\ of \\
&\quad\ Pls \Rightarrow pad\ k\ (bin.Pl\ BIT\ True) \\
&\quad\ | Min \Rightarrow pad\ k\ (bin.Min\ BIT\ False) \\
&\quad\ | (w\ BIT\ b) \Rightarrow (pad\ k\ w)\ BIT\ b)
\end{aligned}$$

The function `get_sign` yields the sign (`Pls` or `bin.Min`) of the binary number passed as argument.

**consts**

$$get\_sign :: bin \Rightarrow bin$$
**primrec**

$$\begin{aligned}
get\_sign-Pls &: get\_sign\ bin.Pl\ =\ bin.Pl \\
get\_sign-Min &: get\_sign\ bin.Min\ =\ bin.Min \\
get\_sign-BIT &: get\_sign\ (w\ BIT\ b)\ =\ (get\_sign\ w)
\end{aligned}$$

The function `trunc` cuts off `n` bits from the right of the bitstring. It is equivalent to right shift with sign extension.

**consts**

$$trunc :: [bin, nat] \Rightarrow bin$$
**primrec**

$$\begin{aligned}
trunc-0 &: trunc\ b\ 0 = b \\
trunc-Suc &: trunc\ b\ (Suc\ n) = \\
&\quad (case\ b\ of \\
&\quad\ Pls \Rightarrow bin.Pl \\
&\quad\ | Min \Rightarrow bin.Min \\
&\quad\ | (w1\ BIT\ b1) \Rightarrow (trunc\ w1\ n))
\end{aligned}$$

The function `trim` `b` `n` keeps the lowest `n` bits of `b` and also keeps the sign.

**consts**

$$trim :: [bin, nat] \Rightarrow bin$$
**primrec**

$$\begin{aligned}
trim-0 &: trim\ b\ 0 = \\
&\quad (case\ b\ of \\
&\quad\ Pls \Rightarrow bin.Pl \\
&\quad\ | Min \Rightarrow bin.Min \\
&\quad\ | (w1\ BIT\ b1) \Rightarrow (get\_sign\ w1)) \quad \text{--- throw away the rest, keep only the} \\
&\quad \text{sign}
\end{aligned}$$

$$\begin{aligned}
trim-Suc &: trim\ b\ (Suc\ n) = \\
&\quad (case\ b\ of \\
&\quad\ Pls \Rightarrow bin.Pl \\
&\quad\ | Min \Rightarrow bin.Min \\
&\quad\ | (b1\ BIT\ w) \Rightarrow (trim\ b1\ n)\ BIT\ w)
\end{aligned}$$

False is equivalent to 0 in the Java bit representation.

The shift operators are not properly described in the JLS (§15.19) either. It is especially unclear what happens if the right-hand-side operand of the shift operators is negative.

The function `shleft` shifts the binary number left `n` bits, by inserting zeroes to the right of the number.

**consts**

*shleft* :: [bin,nat] ⇒ bin

**primrec**

*shleft-0* : shleft b 0 = b

*shleft-Suc* : shleft b (Suc n) = (shleft b n) BIT False

The function `insNull b n` inserts an infinite number of False's at position `n` of the bitstring, counted from the right.

**consts**

*insNull* :: [bin,nat] ⇒ bin

**primrec**

*insNull-0* : insNull b 0 = bin.Pls

*insNull-Suc* : insNull b (Suc n) =  
     (case b of  
       Pls ⇒ bin.Pls  
      | Min ⇒ bin.Pls  
      | (b1 BIT w) ⇒ (insNull b n) BIT w )

**consts**

*map-bin* :: [bool⇒bool, bin] ⇒ bin

**primrec**

*map-bin-Pls* : map-bin f bin.Pls = (if f False then bin.Min else bin.Pls)

*map-bin-Min* : map-bin f bin.Min = (if f True then bin.Min else bin.Pls)

*map-bin-BIT* : map-bin f (w BIT x) =  
     (map-bin f w) BIT (f x)

The function `zip_bin` joins two bitstrings into one. The function passed as argument calculates the bits and thus determines which bitwise operation is performed on the bitstrings. We **do not** assume that both arguments are padded.

**consts**

*zip-bin* :: [[bool,bool]⇒bool,bin,bin] ⇒ bin

**primrec**

*zip-bin-Pls* : zip-bin f-bool x bin.Pls =

    (case x of  
       Pls ⇒ if (f-bool False False) then bin.Min else bin.Pls



```

| Min ⇒ if (f-bool True False) then bin.Min else bin.Pls
| (w BIT b) ⇒ (map-bin (λx. f-bool x False) w)
                BIT (f-bool b False))
zip-bin-Min : zip-bin f-bool x bin.Min =
  (case x of
    Pls ⇒ if (f-bool False True) then bin.Min else bin.Pls
  | Min ⇒ if (f-bool True True) then bin.Min else bin.Pls
  | (w BIT b) ⇒ (map-bin (λx. f-bool x True) w)
                BIT (f-bool b True))
zip-bin-BIT : zip-bin f-bool x (w2 BIT b2) =
  (case x of
    Pls ⇒ (map-bin (f-bool False) w2) BIT (f-bool False b2)
  | Min ⇒ (map-bin (f-bool True) w2) BIT (f-bool True b2)
  | (w1 BIT b1) ⇒
    (zip-bin f-bool w1 w2) BIT (f-bool b1 b2) )

```

## 17.1 Bitwise Operators - AND, OR, XOR

This section formalizes the bitwise AND, OR, and exclusive OR operators.

Java Language Specification [GJSB00], §15.22, 15.22.1

“The bitwise operators [...] include the AND operator &, exclusive OR operator ^, and inclusive OR operator |.<sup>(1)</sup> [...] Each operator is commutative if the operand expressions have no side effects. Each operator is associative.<sup>(2)</sup> [...] For &, the result value is the bitwise AND of the operand values. For ^, the result value is the bitwise exclusive OR of the operand values. For |, the result value is the bitwise inclusive OR of the operand values. For example, the result of the expression 0xff00 & 0xf0f0 is 0xf000. The result of 0xff00 ^ 0xf0f0 is 0x0ff0. The result of 0xff00 | 0xf0f0 is 0xffff.<sup>(3)</sup>”

These bitwise operators are formalized as follows:

### constdefs

```

JavaInt-bitand :: [JavaInt,JavaInt] ⇒ JavaInt  (infixl &# 60)
x &# y ≡
  number-of (zip-bin (op &::[bool,bool]⇒bool) (bin-of x) (bin-of y))

JavaInt-bitxor :: [JavaInt,JavaInt] ⇒ JavaInt  (infixl ^# 59)
x ^# y ≡
  number-of (zip-bin (λ (x::bool) y. x ≠ y) (bin-of x) (bin-of y))

JavaInt-bitor :: [JavaInt,JavaInt] ⇒ JavaInt  (infixl |# 58)
x |# y ≡
  number-of (zip-bin (op |::[bool,bool]⇒bool) (bin-of x) (bin-of y))

```

where `bin-of` transforms a `Javalnt` into its bitstring representation, `zip-bin` merges two bitstrings into one by applying a function (which is passed as the first argument) to each bit pair in turn, and `number-of` turns the resulting bitstring back into a `Javalnt`.

```

lemma trim-True [simp] :
  ((number-of(bin.Pls BIT False BIT x))::int) = number-of(bin.Pls BIT x)
apply (simp add: number-of-def)
apply (cases x)
apply (auto)
apply (simp-all only: iszero-def)
done

```

```

lemma trim-False [simp] :
  ((number-of(bin.Min BIT True BIT x))::int) = number-of(bin.Min BIT x)
apply (simp add: number-of-def)
apply (cases x)
apply (auto)
apply (simp-all only: iszero-def)
done

```

The commutativity of the three operators is first proven generically, which makes the concrete proofs for the three operators trivial:

```

lemma zip-bin-commute:
  ( $\bigwedge x y. ((f x y) = (f y x)) \implies ((zip\text{-}bin\ f\ a\ b) = (zip\text{-}bin\ f\ b\ a))$ )
proof –
  have * : ( $\bigwedge x y. (f x y) = (f y x) \implies ((\lambda x. f x\ False) = f\ False)$ )
  by (rule ext, auto)
  moreover
  have **: ( $\bigwedge x y. (f x y) = (f y x) \implies ((\lambda x. f x\ True) = f\ True)$ )
  by (rule ext, auto)

assume 1: ( $\bigwedge x y. ((f x y) = (f y x))$ )
with 1 show ((zip-bin f a b) = (zip-bin f b a))
  apply (rule-tac x = b in spec)
  apply (induct rule: bin.induct) — induction over a

  apply (simp-all add: * ** split add: bin.split)
  done
qed

```

```

lemma bitand-commute:
   $a \&\# b = b \&\# a$ 
by (simp add: JavaInt-bitand-def, subst zip-bin-commute, auto)

```

```

lemma bitxor-commute:
   $a \hat{\#} b = b \hat{\#} a$ 
by (simp add: JavaInt-bitxor-def, subst zip-bin-commute, auto)

```

```

lemma bitor-commute:
   $a \# b = b \# a$ 
by (simp add: JavaInt-bitor-def, subst zip-bin-commute, auto)

```

end

## 18 Formalizing the Exceptional Behavior Java Integers

**theory** *JavaIntegers* = *JavaIntegersDiv* + *Lifting*:

The Java Language Specification introduces the concept of *exception* in expressions and statements of the language:

Java Language Specification [GJSB00], §11.3, §11.3.1

“The control transfer that occurs when an exception is thrown causes abrupt completion of expressions (§15.6) and statements (§14.1) until a catch clause is encountered that can handle the exception [...]  
when the transfer of control takes place, all effects of the statements executed and expressions evaluated before the point from which the exception is thrown must appear to have taken place. No expressions, statements, or parts thereof that occur after the point from which the exception is thrown may appear to have been evaluated.”

Thus, exceptions have two aspects in Java:

- they change the control flow of a program,
- they are a particular kind of side-effect (i.e. an exception object is created), and they prevent program parts from having side-effects.

While we deliberately neglect the latter aspect in our model (which can be handled in a Hoare Calculus on full Java, for example, when integrating our expression language into the statement language), we have to cope with the former aspect since it turns out to have dramatic consequences for the rules over Java expressions (these effects have not been made precise in the JLS). So far, our normal behavior model is a completely denotational model; each expression is assigned a value by our semantic definitions. We maintain this denotational view, with the consequence that we have to introduce *exceptional values* that are assigned to expressions that “may [not] appear to have been evaluated”. In the language fragment we are considering, only one kind of exception may occur:

“The only numeric operators that can throw an exception (§11) are the integer divide operator / (§15.17.2) and the integer remainder operator % (§15.17.3), which throw an ArithmeticException if the right-hand operand is zero.”

In order to achieve a clean separation of concerns, we apply the technique developed in [BW03]. Conceptually, a theory morphism is used to convert a normal behavior model into a model enriched by exceptional behavior. Technically, the effect is achieved by redefining all operators such as  $+$ ,  $-$ ,  $*$  etc. using “semantical wrapper functions” and the normal behavior definitions given in the previous chapters. Two types of theory morphisms can be distinguished: One for a *one-exception world*, the other for a *multiple-exception world*. While the former is fully adequate for the arithmetic language fragment we are discussing throughout this report, the latter is the basis for future extensions by e.g. array access constructs which may raise *out-of-bounds exceptions*. In the following, we therefore present the former in more detail and only outline the latter.

## 18.1 The One-Exception Theory Morphism

We begin with the introduction of a type constructor that disjointly adds to a type  $\alpha$  a failure element such as  $\perp$  (see e.g. [Win93], where the following construction is also called “lifting”). We declare a type class *bot* for all types containing a failure element  $\perp$  and define as *semantical combinator*, i.e. as “wrapper function” of this theory morphism, the combinator *strictify* that turns a function into its *strict extension* wrt. the failure elements:

*strictify*  $:: [ 'a \Rightarrow 'b, 'a ] \Rightarrow 'b$   
*strictify*  $?f ?x \equiv \text{if } ?x = UU \text{ then } UU \text{ else } ?f ?x$

Moreover, we introduce the definedness predicate  $DEF :: 'a \Rightarrow \text{bool}$  by  $DEF ?x \equiv ?x \neq UU$ .

Now we introduce a concrete type constructor that lifts any type  $\alpha$  into the type class *bot*:

**datatype**  $\text{up}(\alpha) = \lfloor \_ \rfloor \alpha \mid \perp$

In the sequel, we write  $t_\perp$  instead of  $\text{up}(t)$ . We define the inverse to the constructor  $\lfloor \_ \rfloor$  as  $\lceil \_ \rceil$ . Based on this infrastructure, we can now define the type *java<sub>i</sub>nt* that includes a failure element:

**types** *java-int* = *JavaInt up*

**arithes** *up* :: (*zero*) *zero*  
**arithes** *up* :: (*one*) *one*

**defs (overloaded)**  
*zero-def* : 0 ≡ [ 0 ]  
*one-def* : 1 ≡ [ 1 ]

**constdefs**  
*minint* :: *java-int*  
*minint* ≡ [ *MinInt* ]  
*maxint* :: *java-int*  
*maxint* ≡ [ *MaxInt* ]

**arithes** *up* :: (*plus*) *plus*  
**arithes** *up* :: (*minus*) *minus*  
**arithes** *up* :: (*times*) *times*  
**arithes** *up* :: (*Divides.div*) *Divides.div*

Furthermore, we can now define the operations on this enriched type; e.g. we convert the *JavaInt* unary minus operator into the related *java-int* operator (note that Isabelle supports overloading):

**defs (overloaded)**  
*plus-def* : *op* + ≡ *strictify* (λ *X*.  
*strictify* (λ *Y*.  
[ (*Lifting.drop* *X*) + (*Lifting.drop* *Y*) ] ) )

**defs (overloaded)**  
*uminus-def* : *uminus* ≡ *strictify* (λ *y*. [ ( - (*Lifting.drop* *y*) ) ] )

**defs (overloaded)**  
*minus-def* : *op* - ≡ *strictify* (λ *X*.  
*strictify* (λ *Y*.  
[ (*Lifting.drop* *X*) - (*Lifting.drop* *Y*) ] ) )

**defs (overloaded)**  
*abs-def* : *abs* ≡ *strictify*(λ*X*. [ *abs* (*Lifting.drop* *X*) ] )

**defs (overloaded)**  
*times-def* : *op* \* ≡ *strictify* (λ *X*.  
*strictify* (λ *Y*.  
[ (*Lifting.drop* *X*) \* (*Lifting.drop* *Y*) ] ) )

**defs (overloaded)**

```
div-def : op div ≡ strictify (λ X.  
  strictify (λ Y.  
    if (Lifting.drop Y) = 0 then UU  
    else [(Lifting.drop X) div (Lifting.drop Y)]))
```

**defs (overloaded)**

```
mod-def : op mod ≡ strictify (λ X.  
  strictify (λ Y.  
    if (Lifting.drop Y) = 0 then UU  
    else [(Lifting.drop X) mod (Lifting.drop Y)]))
```

All binary arithmetic operators that are strict extensions like  $-$  or  $*$  are constructed analogously; the equality and the logical operators like the strict logical AND  $\&$  follow this scheme as well. For the division and modulo operators  $/$  and  $\%$ , we add case distinctions whether the divisor is zero (yielding  $\perp$ ). Java's non-strict logical AND  $\&\&$  is defined in our framework by explicit case distinctions for  $\perp$ .

This adds new rules like  $X + \perp = \perp$  and  $\perp + X = \perp$ .

## 18.2 Proof of Basic Definedness

**lemma zero-DEF[simp] : DEF (0::java-int)**  
by (simp add: zero-def)

**lemma one-DEF[simp] : DEF (1::java-int)**  
by (simp add: one-def)

**lemma maxint-DEF[simp] : DEF (maxint)**  
by (simp add: maxint-def)

**lemma minint-DEF[simp] : DEF (minint)**  
by (simp add: minint-def)

## 18.3 Proof of exception propagation rules

**lemma uminus-UU[simp] : -(UU::('a::minus)up) = UU**  
by (simp add: uminus-def)

**lemma minus-UU1[simp] : (UU - (X::('a::minus)up)) = UU**  
by (simp add: minus-def)

**lemma minus-UU2[simp] : ((X::('a::minus)up) - UU) = UU**  
by (simp add: minus-def)

**lemma add-UU1[simp] : (UU + (X::('a::plus)up)) = UU**  
by (simp add: plus-def)

**lemma** *add-UU2[simp]* :  $((X::('a::plus)up) + UU) = UU$   
**by** (*simp add: plus-def*)

**lemma** *times-UU1[simp]* :  $(UU * (X::('a::times)up)) = UU$   
**by** (*simp add: times-def*)

**lemma** *times-UU2[simp]* :  $((X::('a::times)up) * UU) = UU$   
**by** (*simp add: times-def*)

**lemma** *div-UU1[simp]* :  $(UU \text{ div } (X::('a::\{Divides.div,zero\})up)) = UU$   
**by** (*simp add: div-def*)

**lemma** *div-UU2[simp]* :  $((X::('a::\{Divides.div,zero\})up) \text{ div } UU) = UU$   
**by** (*simp add: div-def*)

**lemma** *mod-UU1[simp]* :  $(UU \text{ mod } (X::('a::\{Divides.div,zero\})up)) = UU$   
**by** (*simp add: mod-def*)

**lemma** *mod-UU2[simp]* :  $((X::('a::\{Divides.div,zero\})up) \text{ mod } UU) = UU$   
**by** (*simp add: mod-def*)

**lemma** *div-0[simp]* :  $((X::('a::\{Divides.div,zero\})up) \text{ div } 0) = UU$   
**by** (*case-tac DEF X,auto,simp add: div-def zero-def*)

**lemma** *mod-0[simp]* :  $((X::('a::\{Divides.div,zero\})up) \text{ mod } 0) = UU$   
**by** (*case-tac DEF X,auto,simp add: mod-def zero-def*)

## 18.4 Lifting Axiomatic Classes to Versions with Exceptional Behaviour

But what happens with the properties established for the normal behavior semantics? They can also be lifted, and this process can even be automated (see [BW03] for details). Thus, the commutativity and associativity laws for normal behavior, e.g.  $(X::JavaInt) + (Y::JavaInt) = Y + X$ , can be lifted to  $(X::JavaInt\ up) + (Y::JavaInt\ up) = Y + X$  by generic proof procedure establishing the case distinctions for failures. However, this works smoothly only if all variables occur on both sides of the equation; variables only occurring on one side have to be restricted to be defined. Consequently, the lifted version of the division theorem requires definedness of the variable that only occurs on one side of the equation; see below.

**lemma** *JavaInt-add-commute2*:  
 $(X::('a::plus-ac0)up) + Y = Y + X$   
**apply** (*case-tac DEF X, case-tac DEF Y,auto*)  
**apply** (*simp add: DEF-X-up plus-def plus-ac0.commute*)  
**done**

**lemma** *JavaInt-add-assoc2*:  
 $(X::('a::plus-ac0)up) + Y + Z = X + (Y + Z)$   
**apply** (*case-tac DEF X, case-tac DEF Y, case-tac DEF Z, auto*)  
**apply** (*simp add: DEF-X-up plus-def plus-ac0.assoc*)  
**done**

**lemma** *JavaInt-add-zero2*:  
 $0 + (X::('a::plus-ac0)up) = X$   
**apply** (*case-tac DEF X, auto*)  
**apply** (*simp add: plus-def zero-def plus-ac0.zero DEF-X-up*)  
**apply** (*auto*)  
**done**

**lemma** *JavaInt-minus-def2*:  
 $(X::('a::ring)up) - Y = X + (- Y)$   
**apply** (*case-tac DEF X, case-tac DEF Y, auto*)  
**apply** (*simp add: DEF-X-up plus-def minus-def uminus-def*)  
**done**

**lemma** *JavaInt-l-neg2*:  
 $DEF(X::('a::ring)up) \implies (-X) + X = 0$   
**apply** (*simp add: DEF-X-up plus-def uminus-def zero-def ring.l-neg*)  
**done**

**lemma** *JavaInt-times-unit2*:  
 $1 * (X::('a::ring)up) = X$   
**apply** (*case-tac DEF X, auto*)  
**apply** (*simp add: times-def one-def DEF-X-up*)  
**apply** (*auto*)  
**done**

**lemma** *JavaInt-times-commute2*:  
 $(X::('a::ring)up) * Y = Y * X$   
**apply** (*case-tac DEF X, case-tac DEF Y, auto*)  
**apply** (*simp add: DEF-X-up times-def ring.m-comm*)  
**done**

**lemma** *JavaInt-times-assoc2*:  
 $(X::('a::ring)up) * Y * Z = X * (Y * Z)$   
**apply** (*case-tac DEF X, case-tac DEF Y, case-tac DEF Z, auto*)  
**apply** (*simp add: DEF-X-up times-def ring.m-assoc*)  
**done**

**lemma** *JavaInt-l-distr2*:  
 $((X::('a::ring)up) + Y) * Z = X * Z + Y * Z$   
**apply** (*case-tac DEF X, case-tac DEF Y, case-tac DEF Z, auto*)  
**apply** (*simp add: times-def plus-def ring.l-distr DEF-X-up*)



**done**

```
instance up :: (plus-ac0)plus-ac0
  apply intro-classes
  apply (auto intro: JavaInt-add-commute2 JavaInt-add-assoc2 JavaInt-add-zero2)
  done
```

```
instance up :: (power)power ..
```

```
defs (overloaded)
```

```
  power-def : (a::('a::power) up) ^ n ≡
    (strictify (λ X. [ (Lifting.drop X) ^ n ] ) a)
```

```
instance up :: (inverse)inverse ..
```

When trying to add the lifted type to the axiomatic class *ring*, we discover a problem: technically, *'a up* is not a ring. The main problem is `l_neg`:  $-(a::'a\ up) + a = (0::'a\ up)$  which simply does not hold in an exception behaviour class, only a weaker form:  $DEF\ (a::'a\ up) \implies -a + a = (0::'a\ up)$  (`JavaInt_l_neg2`). Similar problems arise for  $(a + b = a + c) = (b = c)$  and  $(b + a = c + a) = (b = c)$  which hold in Isabelle 2004 for the axiomatic class *semiring*, and

$(a::'a\ up)\ dvd\ (1::'a\ up) \implies inverse\ a = (0::'a\ up)$  which did not make it into Isabelle04.

There is a big advantage for Isabelle2004 here: the organization into classes is much more fine-grained and therefore the damage is smaller. In Isabelle 2004, this structure still fits in “almost\_semiring”.

```
lemma DEF-times [simp] : DEF((X::java-int) * Y) = (DEF X & DEF Y)
  apply (cases DEF X, cases DEF Y, auto simp: zero-def)
  apply (simp-all add: times-def)
  done
```

```
lemma DEF-plus [simp] : DEF((X::java-int) + Y) = (DEF X & DEF Y)
  apply (cases DEF X, cases DEF Y, auto simp: zero-def)
  apply (simp-all add: plus-def)
  done
```

```
lemma DEF-minus [simp] : DEF((X::java-int) - Y) = (DEF X & DEF Y)
  apply (cases DEF X, cases DEF Y, auto simp: zero-def)
  apply (simp-all add: minus-def)
  done
```

```
lemma DEF-uminus [simp] : DEF(-(X::java-int)) = (DEF X)
  apply (cases DEF X, auto simp: zero-def)
```

```

apply (simp-all add: uminus-def)
done

```

```

lemma DEF-div-EQ [simp] : DEF((X::java-int) div Y) = (DEF X & DEF Y &
Y ≠ 0)
apply (cases DEF X, cases DEF Y, auto simp: zero-def)
apply (simp-all add: div-def)
apply (auto simp: DEF-X-up)
done

```

```

lemma DEF-mod-EQ [simp] : DEF((X::java-int) mod Y) = (DEF X & DEF Y
& Y ≠ 0)
apply (cases DEF X, cases DEF Y, auto simp: zero-def)
apply (simp-all add: mod-def)
apply (auto simp: DEF-X-up)
done

```

## 18.5 Lifting Number Representations and Bridging to Computations

```

instance up :: (number)number ..

```

```

defs (overloaded)
  number-of-def : (number-of:: bin ⇒ (('a::number)up)) b ≡ [ (number-of b)
]

```

Computation rules:

```

lemma uminus-compute[simp] : -(number-of X) = (?X::java-int)
by (simp add: uminus-def number-of-def)

```

```

lemma add-compute[simp] : (number-of X) + (number-of Y) = (?X::java-int)
by (simp add: plus-def number-of-def)

```

```

lemma times-compute[simp] : (number-of X) * (number-of Y) = (?X::java-int)
by (simp add: times-def number-of-def)

```

```

lemma minus-compute[simp] : (number-of X) - (number-of Y) = (?X::java-int)
by (simp add: minus-def number-of-def)

```

```

lemma div-compute[simp] : (number-of X) div (number-of Y) =
  (if ((number-of Y - MinInt-int) mod -
      (2 * MinInt-int) + MinInt-int) = 0
   then UU::java-int
   else [ (number-of X) div (number-of Y) ])
apply(simp only: div-def number-of-def Int-to-JavaInt-def
      JavaIntegersDef.zero-def strict22-DEF DEF-lift drop-lift)

```

```

apply(subgoal-tac (number-of  $Y = \text{Abs-JavaInt } (0::\text{int}) =$ 
  ( $(\text{number-of } Y - \text{MinInt-int}) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int}$ 
 $= 0$ ))
apply simp
apply (simp add: JavaIntegersDef.number-of-def Int-to-JavaInt-def)
apply (subst Abs-JavaInt-inject)
apply (simp-all)
apply (insert Int-to-JavaInt-charn)
apply (simp add: Int-to-JavaInt-def Rep-Abs-remove-general )
done

```

```

lemma mod-compute[simp] : (number-of  $X$ ) mod (number-of  $Y$ ) =
  (if (( $(\text{number-of } Y - \text{MinInt-int}) \bmod -$ 
    ( $2 * \text{MinInt-int}) + \text{MinInt-int} = 0$ 
    then  $UU::\text{java-int}$ 
    else  $\lfloor (\text{number-of } X) \bmod (\text{number-of } Y) \rfloor$  )
apply(simp only: mod-def number-of-def Int-to-JavaInt-def
  JavaIntegersDef.zero-def strict22-DEF DEF-lift drop-lift)
apply(subgoal-tac (number-of  $Y = \text{Abs-JavaInt } (0::\text{int}) =$ 
  ( $(\text{number-of } Y - \text{MinInt-int}) \bmod - (2 * \text{MinInt-int}) + \text{MinInt-int}$ 
 $= 0$ ))
apply simp
apply (simp add: JavaIntegersDef.number-of-def Int-to-JavaInt-def)
apply (subst Abs-JavaInt-inject)
apply (simp-all)
apply (insert Int-to-JavaInt-charn)
apply (simp add: Int-to-JavaInt-def Rep-Abs-remove-general )
done

```

```

lemma ex1:  $5 \text{ div } 3 = (1::\text{java-int})$ 
apply (simp add: div-mod-example2 one-def MinInt-int-def BitLength-def)
done

```

## 18.6 Lifting JavaInteger Theorems

```

lemma JavaInt-div-mod-result :
 $\llbracket \text{DEF } Y; Y \neq 0 \rrbracket \implies ((X::\text{java-int}) \text{ div } Y) * Y + (X \bmod Y) = X$ 
apply (case-tac DEF X, auto)
apply (simp add: DEF-X-up div-def mod-def times-def plus-def zero-def, auto)
apply (subst JavaInt-times-commute)
apply (rule JavaInt-div-mod-result)
done

```

```

lemma JavaInt-0-div[simp]:  $\llbracket \text{DEF } X; X \neq 0 \rrbracket \implies 0 \text{ div } X = (0::\text{java-int})$ 
by (simp add: DEF-X-up div-def mod-def times-def plus-def zero-def, auto)

```

```

lemma minint-div-minusone[simp]: minint div -1 = minint
  apply (simp add: DEF-X-up div-def number-of-def
          minint-def MinInt-div-minusone)
  apply (simp add: JavaIntegersDef.number-of-def JavaIntegersDef.zero-def)
  apply (subst Int-to-JavaInt-ident)
  defer 1
  apply (subst Abs-JavaInt-inject)
  apply (simp-all)
  done

lemma diff-self[simp]:  $\llbracket \text{DEF } a; a \neq 0 \rrbracket \implies a \text{ div } a = (1::\text{java-int})$ 
  by (simp add: DEF-X-up div-def zero-def one-def, auto)

lemma mod-self[simp]:  $\llbracket \text{DEF } a; a \neq 0 \rrbracket \implies a \text{ mod } a = (0::\text{java-int})$ 
  by (simp add: DEF-X-up mod-def zero-def one-def, auto)

lemma mod-divides[simp]:
  assumes 1: DEF a
  assumes 2: DEF b
  assumes 3: (Rep-JavaInt(Lifting.drop a) mod (Rep-JavaInt (Lifting.drop b))
  = 0
  — i.e. representation a divides representation b
  assumes 4: b ≠ 0
  shows a mod b = (0::java-int)
  apply (insert 1 2 3 4)
  apply (simp add: DEF-X-up mod-def zero-def one-def, auto)
  apply (rule JavaIntegersDiv.JavaInt-mod-0, auto)
  done

lemma minint-mod-minusone[simp]: minint mod -1 = 0
  apply (simp add: DEF-X-up mod-def minint-def zero-def number-of-def one-def, auto)
  apply (simp add: JavaIntegersDef.number-of-def JavaIntegersDef.zero-def)

  apply (rule swap [of Int-to-JavaInt (-1::int) = Abs-JavaInt (0::int)])
  apply (subst Int-to-JavaInt-ident)
  prefer 2
  apply (subst Abs-JavaInt-inject, auto)
  done

lemma div-neg-shift:
  assumes 1: a ≠ minint
  assumes 2: b ≠ minint

```

```

shows    a div (-b) = (-a) div b
apply (case-tac DEF a,auto)
apply (case-tac DEF b,auto)
apply (insert 1 2)
apply (simp add: DEF-X-up div-def minint-def zero-def
        number-of-def one-def uminus-def JavaInt-div-neg-shift,auto)
apply (rule JavaInt-div-neg-shift, auto)
done

```

```

lemma mod-neg1:
  assumes 1:a≠minint
  assumes 2:b≠minint
  shows   -(a mod b) = (-a) mod b
  apply (case-tac DEF a,auto)
  apply (case-tac DEF b,auto)
  apply (insert 1 2)
  apply (simp add: DEF-X-up mod-def minint-def zero-def
        number-of-def one-def uminus-def, auto)

  apply(rule JavaInt-mod-neg1,auto)
done

```

```

lemma mod-neg2:
  shows   a mod (-b) = (a::java-int) mod b
  apply (case-tac DEF a,auto)
  apply (case-tac DEF b,auto)
  apply (simp add: DEF-X-up mod-def minint-def zero-def
        number-of-def one-def uminus-def, auto)

  apply(rule JavaInt-mod-neg2)
done

```

## 18.7 The Multiple-Exception Theory Morphism

We briefly consider the case of a semantics with several different exceptions, e.g. array access which can possibly lead to out-of-bounds exceptions. Such a change of the model can be achieved by exchanging the theory morphism, leaving the normal behavior model unchanged.

It suffices to present the differences to the previous theory morphism here. Instead of the class `bot` we introduce the class `exn` requiring a family of undefined values  $\perp_e$ . The according type constructor is defined as:

**datatype** up( $\alpha$ ) =  $[_]$   $\alpha$  |  $\perp$  exception

and analogously to  $[_]$  we define  $\text{exn-of}(\perp_e) = e$  as the inverse of the constructor  $\perp$ ;  $\text{exn-of}$  is defined by an arbitrary but fixed HOL-value *arbitrary* for  $\text{exn-of}(\perp_e) = \text{arbitrary}$ . Definedness is  $\text{DEF}(x) = (\forall e. x \neq \perp_e)$ .

The definition of operators is analogous to the previous section for the canonical cases; and the resulting lifting as well. Note, however, that the lifting of the commutativity laws fails and has to be restricted to the following:

$$\begin{aligned} & \llbracket \text{DEF } X = \text{DEF } Y \wedge \text{exn-of } X = \text{exn-of } Y \rrbracket \\ & \implies (X :: \text{JAVAINT}) + Y = Y + X \end{aligned}$$

These restrictions caused by the lifting reflect the fact that commutativity does not hold in a multi-exception world; if the expression on the left-hand side does not raise the same exception as the one on the right-hand side, the expression order cannot be changed.

Of course, this problem could also be overcome by defining a congruence that identifies all exceptions; however, this complicates matters substantially.

We refrain from such a refined model considering different exceptions in this work. However, we emphasize that our proposed technique to use a theory morphism not only leads to a clear separation of concerns in the semantic description of Java, but also to the systematic introduction of the side-conditions of arithmetic laws in Java that are easily overlooked.

**end**

**theory** *JavaArith* = *JavaIntegers* + *Lifting* :

**types** *java-bool* = *bool up*

**constdefs**

*TRUE*            :: *java-bool*  
*TRUE*            ≡ [ *True* ]

*FALSE*           :: *java-bool*  
*FALSE*           ≡ [ *False* ]

**lemma** *DEF-TRUE* [*simp*]: *DEF TRUE*  
**by** (*auto simp:TRUE-def*)

**lemma** *DEF-FALSE* [*simp*]: *DEF FALSE*

by (auto simp:FALSE-def)

**constdefs**

*NOT*        :: java-bool  $\Rightarrow$  java-bool  
*NOT*         $\equiv$  strictify(Lifting.lift o Not o Lifting.drop)  
*AND*        :: [java-bool, java-bool]  $\Rightarrow$  java-bool  
*AND*         $\equiv$  strictify ( $\lambda X.$   
              strictify ( $\lambda Y.$   
                        [ (Lifting.drop X) & (Lifting.drop Y) ] ))  
  
*NEG*        :: java-int  $\Rightarrow$  java-bool  
*NEG*         $\equiv$  strictify ( $\lambda X.$  [ neg (Rep-JavaInt(Lifting.drop X)) ] )  
  
*LE*         :: [java-int, java-int]  $\Rightarrow$  java-bool  
*LE*          $\equiv$  strictify ( $\lambda X.$   
              strictify ( $\lambda Y.$   
                        [ (Lifting.drop X)  $\leq$  (Lifting.drop Y) ] ))  
  
*LESS*       :: [java-int, java-int]  $\Rightarrow$  java-bool  
*LESS*        $\equiv$  strictify ( $\lambda X.$   
              strictify ( $\lambda Y.$   
                        [ (Lifting.drop X) < (Lifting.drop Y) ] ))

**lemma** *NOT-UU[simp]* : (NOT UU) = UU  
by (simp add: NOT-def)

**lemma** *NEG-UU[simp]* : (NEG UU) = UU  
by (simp add: NEG-def)

**lemma** *abs-UU[simp]* : (abs (UU::java-int)) = UU  
by (simp add: abs-def)

**lemma** *AND-UU1[simp]* : (AND UU X) = UU  
by (simp add: AND-def)

**lemma** *AND-UU2[simp]* : (AND X UU) = UU  
by (simp add: AND-def)

**lemma** *LE-UU1[simp]* : (LE UU X) = UU  
by (simp add: LE-def)

**lemma** *LE-UU2[simp]* : (LE X UU) = UU  
by (simp add: LE-def)

**lemma** *LESS-UU1*[simp] : (LESS UU X) = UU  
by (simp add: LESS-def)

**lemma** *LESS-UU2*[simp] : (LESS X UU) = UU  
by (simp add: LESS-def)

A bit of strict logic ...

**lemma** *DEF-isTRUE*[simp] : X = TRUE  $\implies$  DEF X  
by simp

**lemma** *DEF-isFALSE*[simp] : X = FALSE  $\implies$  DEF X  
by simp

**lemma** *DEF-isTRUE-I* [intro]:  $\llbracket$  DEF X; Lifting.drop X  $\rrbracket \implies X = TRUE$   
by(auto simp: DEF-X-up TRUE-def)

**lemma** *DEF-isFALSE-I* :  $\llbracket$  DEF X;  $\sim$  Lifting.drop X  $\rrbracket \implies X = FALSE$   
by(auto simp: DEF-X-up FALSE-def)

**lemma** *DEF-LE-EQ* [simp] : DEF(LE X Y) = (DEF X & DEF Y)  
apply (cases DEF X, cases DEF Y, auto)  
apply (simp add: LE-def)  
done

**lemma** *DEF-LESS-EQ* [simp] : DEF(LESS X Y) = (DEF X & DEF Y)  
apply (cases DEF X, cases DEF Y, auto)  
apply (simp add: LESS-def)  
done

**lemma** *DEF-AND-EQ* [simp] : DEF(AND X Y) = (DEF X & DEF Y)  
apply (cases DEF X, cases DEF Y, auto)  
apply (simp add: AND-def)  
done

**lemma** *DEF-NEG-EQ* [simp] : DEF(NEG X) = (DEF X)  
by (cases DEF X, auto, simp add: NEG-def)

**lemma** *DEF-NOT-EQ* [simp] : DEF(NOT X) = (DEF X)  
by (cases DEF X, auto, simp add: NOT-def)

**lemma** *DEF-abs-EQ* [simp] : DEF(abs (X::java-int)) = (DEF X)  
by (cases DEF X, auto, simp add: abs-def)



## 18.8 Lifting orders and related theorems.

Here, there are many choices to lift the premises. They can be formalized in terms of the mathematical integers, normal behaviour or exceptional behaviour JavaIntegers. As a guideline, we chose the latter option since one may want to express these entities in terms of a programming logic in Java.

```

lemma quotient-sign-plus :
  assumes 0: DEF d & DEF n
  assumes 1: abs (Rep-JavaInt(Lifting.drop d)) <= abs(Rep-JavaInt(Lifting.drop n))
  assumes 2: neg (Rep-JavaInt (Lifting.drop n)) = neg (Rep-JavaInt (Lifting.drop d))
  assumes 3: n ≠ minint | d ≠ (- 1)
  assumes 4: d ≠ 0
  shows LESS 0 (n div d) = TRUE
  apply (insert 0 1 2 3 4)
  apply (simp add: DEF-X-up div-def zero-def)
  apply (simp-all add: DEF-X-up LESS-def div-def zero-def minint-def TRUE-def)
  apply (auto simp : quotient-sign-plus uminus-def minus-def one-def div-def)
done

```

```

lemma quotient-sign-minus1 :
  assumes 0: d ~ = minint
  assumes 1: LESS d 0 = TRUE
  assumes 2: LESS 0 n = TRUE
  assumes 3: LE (- d) (n) = TRUE
  shows LESS (n div d) 0 = TRUE
  apply (subgoal-tac DEF d & DEF n & DEF (-d))
  defer
  apply (insert 0 1 2 3)
  apply (drule DEF-isTRUE)
  apply (drule DEF-isTRUE)
  apply auto

  apply (rule DEF-isTRUE-I)
  apply (subgoal-tac d ≠ 0)
  apply (simp)
  prefer 2
  apply (auto simp: DEF-X-up LESS-def LE-def div-def zero-def TRUE-def uminus-def)
  apply (rule quotient-sign-minus1)
  apply (drule iffD1 [OF le-quasi-def])
  defer 1
  apply (drule iffD1 [OF le-quasi-def]) back
  defer 1
  apply (subst uminus-homom[symmetric])
  apply (simp-all add: JavaIntegersDef.zero-def minint-def
          JavaIntegersRing.JavaInt-le-def)
done

```

```

lemma quotient-sign-minus2 :
  assumes 1: LESS 0 d = TRUE
  assumes 2: LESS n 0 = TRUE
  assumes 3:  $n \neq \text{minint}$ 
  assumes 4: LE n (- d) = TRUE
  shows LESS (n div d) 0 = TRUE
  apply (subgoal-tac DEF d & DEF n)
  defer
  apply (insert 1 2 3 4)
  apply (drule DEF-isTRUE)
  apply (drule DEF-isTRUE)
  apply auto
  apply (rule DEF-isTRUE-I)
  apply (subgoal-tac  $d \neq 0$ )
  apply (simp)
  prefer 2
  apply (auto simp: DEF-X-up LESS-def div-def zero-def TRUE-def)
  apply (rule quotient-sign-minus2)
  apply (auto simp: minint-def NEG-def LE-def abs-def
    TRUE-def NEG-def less-def zero-def
    uminus-def one-def)
  apply (drule iffD1 [OF le-quasi-def])
  defer 1
  apply (drule iffD1 [OF le-quasi-def]) back
  defer 1
  apply (erule swap)
  apply (rule Rep-JavaInt-inject[THEN iffD1])
  defer 1
  apply (drule le-neg-bound)
  apply (simp-all add: JavaIntegersDef.zero-def MinInt-def)
  done

```

```

lemma neg-mod-bound:
  assumes 1: LESS b 0 = TRUE
  assumes 2:  $b \neq \text{minint}$ 
  assumes 3: DEF a
  shows LESS b (a mod b) = TRUE
  apply (subgoal-tac DEF b)
  defer
  apply (insert 1 2 3)
  apply (drule DEF-isTRUE)
  apply auto
  apply (rule DEF-isTRUE-I)
  apply (subgoal-tac  $b \neq 0$ )
  apply (simp)
  prefer 2

```

```

apply (auto simp: DEF-X-up LESS-def mod-def zero-def TRUE-def)
apply (rule neg-mod-bound)
apply (auto simp: minint-def)
done

```

```

lemma pos-mod-bound:
  assumes 1: LESS 0 b = TRUE
  assumes 2: DEF a
  shows LESS (a mod b) b = TRUE
  apply (subgoal-tac DEF b)
  defer
  apply (insert 1 2)
  apply (drule DEF-isTRUE)
  apply auto
  apply (rule DEF-isTRUE-I)
  apply (subgoal-tac b ≠ 0)
  apply (simp)
  prefer 2
  apply (auto simp: DEF-X-up LESS-def mod-def zero-def TRUE-def)
  apply (rule pos-mod-bound)
  apply (auto)
  done

```

```

lemma neg-mod-sign:
  assumes 1: LESS a 0 = TRUE
  assumes 2: b ≠ 0
  assumes 3: DEF b
  shows LE (a mod b) 0 = TRUE
  apply (subgoal-tac DEF a)
  defer
  apply (insert 1 2 3)
  apply (drule DEF-isTRUE)
  apply auto
  apply (rule DEF-isTRUE-I)
  apply (subgoal-tac b ≠ 0)
  apply (simp)
  prefer 2
  apply (auto simp: DEF-X-up LE-def LESS-def mod-def zero-def TRUE-def)
  apply (rule neg-mod-sign)
  apply (auto)
  done

```

```

lemma pos-mod-sign:
  assumes 1: LE 0 a = TRUE
  assumes 2: b ≠ 0
  assumes 3: b ≠ minint
  assumes 4: DEF b

```

```

shows     $LE\ 0\ (a\ mod\ b) = TRUE$ 
apply (subgoal-tac DEF a)
defer
apply (insert 1 2 3 4)
apply (drule DEF-isTRUE)
apply auto
apply (rule DEF-isTRUE-I)
apply (subgoal-tac  $b \neq 0$ )
apply (simp)
prefer 2
apply (auto simp: DEF-X-up LE-def LESS-def mod-def zero-def TRUE-def)
apply (rule pos-mod-sign)
apply (auto simp: minint-def)
done

```

**lemma** *JavaInt-mod-less*:

```

assumes 1:  $b \neq 0$ 
assumes 2:  $b \neq minint$ 
assumes 3: DEF a
assumes 4: DEF b
shows     $LESS\ (abs\ (a\ mod\ b))\ (abs\ b) = TRUE$ 
apply (insert 1 2 3 4)
apply (rule DEF-isTRUE-I)
apply (simp)
apply (auto simp: DEF-X-up LE-def LESS-def mod-def zero-def TRUE-def
abs-def)
apply (rule JavaInt-mod-less)
apply (auto simp: minint-def)
done

```

**end**

## 19 Conclusions and Future Work

In this report we presented a formalization of Java's two's-complement integral types in Isabelle/HOL. Our formalization includes both normal and exceptional behavior (based on one exception). The normal behaviour Java integers presented here form a ring, and this even extends to the exceptional behaviour model with the only change that the inverse element rule must be weakened to  $DEF(A) \implies A + (-A) = 0$ . Both for the normal and the exceptional behaviour model, a number of properties for the Java division and remainder operation have been proven (and tested against some implementations of the Java compiler). We get a two's-complement representation by redefining the conversion function *number\_of* which is already provided for *int*. This representation can be used for those operators that are defined

bitwise, but this theory has only been developed to a very basic extent. Altogether, the existing Isabelle theories make it relatively easy to achieve standard number-theoretic properties for types that are defined as a subset of the Isabelle/HOL integers.

Such a formalization is a necessary prerequisite for the verification of efficient arithmetic Java programs such as encryption algorithms, in particular in tools like Jive [MPH00] that generate verification conditions over arithmetic formulae from such programs.

Our formalization of the normal behavior is based on a direct analysis of the Java Language Specification [GJSB00] and led to the discovery of several underspecifications and ambiguities (see 5 (4), 16.1, 15.1, 17). These underspecifications are highly undesirable since even compliant Java compilers may interpret the same program differently, leading to unportable code. In the future, we strongly suggest to supplement informal language definitions by machine-checked specifications like the one we present in this report as a part of the normative basis of a programming language.

We applied the technique of mechanized theory morphisms (developed in [BW03]) to our Java arithmetic model in order to achieve a clear separation of concerns between normal and exceptional behavior. Moreover, we showed that the concrete exceptional model can be exchanged — while controlling the exact side-conditions that are imposed by a concrete exceptional model. For the future, this leaves the option to use a lifting to the *exception state monad* [LHJ95] mapping the type `JAVAINT` to state  $\Rightarrow(\text{Javalnt}_{\perp}, \text{state})$  in order to give semantics to expressions with side-effects like `i++ + i`.

Of course, more rules can be added to our theory in order to allow effective automatic computing of large (ground) expressions — this has not been in the focus of our interest so far. With respect to proof automation in `Javalnt`, it is an interesting question whether arithmetic decision procedures of most recent Isabelle versions (based on Cooper’s algorithm for Presburger Arithmetic) can be used to decide analogous formulae based on machine arithmetic. While an *adoption* of these procedures to Java arithmetic seems impossible (this would require cancellation rules such as  $(a \leq b) = (k \times a \leq k \times b)$  for nonnegative  $k$  which do not hold in Java), it is possible to retranslate `Javalnt` formulae to standard integer formulae; remainder sub-expressions can be replaced via  $P(a \bmod b) = \exists m. 0 \leq m < a \wedge (a - m) \mid b \wedge P(m)$ , such that finally a Presburger formula results. Since a translation leads to an exponential blow-up in the number of quantifiers (a critical feature for Cooper’s algorithm), it remains to be investigated to what extent this approach is feasible in practice.

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